Quiz#2, Mathematical Statistics, 2017 Fall [+8 points] +5

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+2 **Q1** [+2] Let $X_1,\ldots,X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$.

(1) Obtain the MLE of $\hat{\lambda}$ (show it is the global maxima).

+1 $L(\lambda|x) = e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i} / \prod_{i=1}^{n} x_i!$

$\ln L(\lambda|x) = -n\lambda + \sum_{i=1}^{n} x_i \ln\lambda - \ln\prod_{i=1}^{n} x_i!$

$\frac{\partial \ln L(\lambda|x)}{\partial\lambda} \Rightarrow -n + \frac{\sum_{i=1}^{n} x_i}{\lambda} \overset{set}{=} 0$

$\hat{\lambda} = \bar{X}$

$\frac{\partial^2 \ln L(\lambda|x)}{\partial^2\lambda} \Rightarrow -\frac{\sum X_i}{\lambda^2} < 0$ ∵ $X \geq 0$

so $\hat{\lambda} = \bar{X}$ is the global maxima

the MLE of $\hat{\lambda}$ is $\bar{X}$

(2) Obtain the MLE $\hat{\eta}$ of the natural parameter η without using the invariance (show it is the global maxima)

+1 $L(\lambda|x) = e^{-n\lambda} e^{\sum x_i \ln\lambda} / \prod_{i=1}^{n} x_i!$

let $\eta = \ln\lambda$, $\lambda = e^{\eta}$

$L(\eta|x) = e^{-ne^{\eta}} e^{\sum x_i \eta} / \prod_{i=1}^{n} x_i!$

$\ln L(\eta|x) = -ne^{\eta} + \sum x_i \eta - \ln\prod x_i!$

$\frac{\partial \ln L(\eta|x)}{\partial\eta} \Rightarrow -ne^{\eta} + \sum X_i \overset{set}{=} 0$

$e^{\eta} = \bar{X}$

$\hat{\eta} = \ln\bar{X}$

$\frac{\partial^2 \ln L(\eta|x)}{\partial^2\eta} \Rightarrow -ne^{\eta} < 0$ ∵ $(\eta > 0)$

so MLE $\hat{\eta}$ is $\ln\bar{X}$

+0 **Q2** [+1] Let $X_1,\ldots,X_n \overset{iid}{\sim} f(x|\beta) = \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right)$, $\beta > 0$, $x > 0$.

Express $f(x|\beta)$ in the form of the natural exponential family. Then, obtain the MLE $\hat{\eta}$ of the natural parameter η (show it is the global maxima)

$L(\beta|x) = \frac{1}{\beta^n} \exp -\frac{\sum X_i}{\beta}$

令 $\eta = -\frac{1}{\beta} > 0$

$\beta = \frac{1}{\eta}$

$L(\eta|x) = \eta^n \exp^{-\sum X_i \eta}$

$\ln L(\eta|x) = n\ln\eta - \sum X_i \eta$

$\frac{\partial \ln L(\eta|x)}{\partial\eta} \Rightarrow \frac{n}{\eta} - \sum X_i \overset{set}{=} 0$

$\eta = \frac{1}{\bar{X}}$

$\frac{\partial^2 \ln L(\eta|x)}{\partial^2\eta} \Rightarrow -\frac{n}{\eta^2} < 0$

so MLE $\hat{\eta} = \frac{1}{\bar{X}}$

+1 **Q3** [+2] Let $X_1,\ldots,X_n \overset{iid}{\sim} N(\mu,\sigma^2)$, where μ is restricted to $\mu \le a$ or $\mu \ge b$ for some numbers $a<b$. Assume that σ^2 is known. Hence, the parameter space is $\Theta=(-\infty,a]\cup[b,\infty)$. Obtain the MLE $\hat{\mu}$ (with some figures to explain it).

$$L(\mu|\sigma^2,x)=\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\frac{\sum(x_i-\mu)^2}{2\sigma^2}\right)$$

$$\ln L(\mu|\sigma^2,x) = -n\ln\sqrt{2\pi}\sigma - \frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}$$

$$\frac{\partial \ln L(\mu|\sigma^2,x)}{\partial\mu} \Rightarrow \frac{\sum(x_i-\mu)}{\sigma^2}=0$$

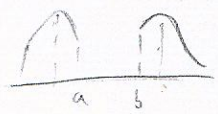
$$\hat{\mu}=\bar{x}$$

$$\frac{\partial^2 \ln L(\mu|\sigma^2,x)}{\partial^2\mu} \Rightarrow -\frac{1}{\sigma^2}<0 \quad \because \sigma^2>0$$

so the MLE of $\hat{\mu}$

$$=\begin{cases} \bar{x} & \text{if } \bar{x}<a \text{ or } \bar{x}>b \\ a & \text{if } \bar{x}=a \\ b & \text{if } \bar{x}=b \\ a \text{ or } b & \text{if } a<\bar{x}<b \end{cases}$$

X



+2 **Q4** [+3] Let $X_1,\ldots,X_n \overset{iid}{\sim} Gamma(\alpha,\beta)$ as in HW#1. Let $\psi(\alpha)=\frac{d}{d\alpha}\log\Gamma(\alpha)$ be the digamma function and $\psi'(\alpha)$ be the trigamma function.

+1 (1) Write down the score functions using the sufficient statistics (T_1,T_2).

$$L(\alpha,\beta|x)=\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \left(\prod_{i=1}^n x_i\right)^{\alpha-1} e^{-\frac{\sum x_i}{\beta}}$$

by factorization theorem (T_1,T_2) is $(\prod_{i=1}^n x_i, \sum x_i)$

$$\ln L(\alpha,\beta|x) = -n\ln\Gamma(\alpha) - n\alpha\ln\beta + (\alpha-1)\ln\prod_{i=1}^n x_i - \sum x_i/\beta$$

$$\frac{\partial\ln L(\alpha,\beta|x)}{\partial\alpha} \Rightarrow -n\psi(\alpha) - n\ln\beta + \ln\prod_{i=1}^n x_i \quad \text{---- (a)}$$

$$\frac{\partial\ln L(\alpha,\beta|x)}{\partial\beta} \Rightarrow -\frac{n\alpha}{\beta}+\frac{\sum x_i}{\beta^2} \quad \text{---- (b)}$$

$S_1(\alpha,\beta)= -n\psi(\alpha) - n\ln\beta + \ln\prod_{i=1}^n x_i$

$S_2(\alpha,\beta)= -\frac{n\alpha}{\beta}+\frac{\sum x_i}{\beta^2}$

Use T_1 and T_2

where $T_1 = \prod_{i=1}^n x_i$ and $T_2 = \sum_{i=1}^n x_i$

+1 (2) Write down the Hessian matrix $H(\alpha,\beta)$.

$$H(\alpha,\beta)=\begin{bmatrix} -n\psi'(\alpha) & -\frac{n}{\beta} \\ -\frac{n}{\beta} & \frac{n\alpha}{\beta^2}-\frac{2\sum x_i}{\beta^3} \end{bmatrix}$$

+0 (3) Let $(\hat{\alpha},\hat{\beta})$ be the solution to $S_1(\alpha,\beta)=S_2(\alpha,\beta)=0$. Write down $H(\hat{\alpha},\hat{\beta})$ in terms of $(\hat{\alpha},\hat{\beta})$.

fit α on (b)

X

$$-\frac{n\alpha}{\beta}+\frac{\sum x_i}{\beta^2} \overset{set}{=} 0$$

$$\frac{\sum x_i}{\beta}=n\alpha$$

$$\hat{\beta}=\frac{\sum x_i}{n\alpha}=\frac{\bar{x}}{\alpha}$$

fit $\hat{\beta}=\frac{\bar{x}}{\alpha}$ on (a)

$$-n\psi(\alpha)-n(\ln\bar{x}-\ln\alpha)+\ln\prod x_i=0$$

$$\ln\alpha = \psi(\alpha)+\ln\bar{x}-\frac{\ln\prod x_i}{n}$$

$$\alpha = e^{\psi(\alpha)+\ln\bar{x}-\frac{\ln\prod x_i}{n}}$$