Quiz#1, Mathematical Statistics, 2017 Fall [5 points] +5

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Let $X_1,\ldots,X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$ and $\mathbf{X}=(X_1,\ldots,X_n)$.

+1 (1) [+1] Write down the pmf of $\mathbf{X}$:

$$f(\mathbf{x}\mid\lambda)=\prod_{i=1}^{n}\frac{e^{-\lambda}\lambda^{x_i}}{x_i!}=\frac{e^{-\lambda n}\lambda^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!}$$ for all $x_i=0,1,2,\ldots$ $\forall i=1,\ldots,n$

+1 (2) [+1] What is the distribution of $T(\mathbf{X})=X_1+\cdots+X_n$?

$T(X)\sim \text{Poisson}(n\lambda)$

+1 (3) [+1] Prove your answer of ~~(1)~~ (2) using the moment generating function

$$M_{T(X)}(t)=E\left(e^{t(X_1+\cdots+X_n)}\right)=E\left(\prod_{i=1}^{n}e^{tX_i}\right)=\prod_{i=1}^{n}E\left(e^{tX_i}\right)\quad \text{(By independent)}$$

$$=\left[E(e^{tX})\right]^n=\left[e^{\lambda(e^t-1)}\right]^n=e^{n\lambda(e^t-1)}\quad \text{—}\langle * \rangle$$

Let $Y\sim \text{Poisson}(n\lambda)$ $\quad M_Y(t)=E[e^{Yt}]=\sum_{y=0}^{\infty}\frac{e^{-\lambda n}(\lambda n)^y}{y!}e^{yt}=\sum_{y=0}^{\infty}\frac{e^{-\lambda n}(\lambda n e^t)^y}{y!}$

$$=e^{n\lambda(e^t-1)}\quad \text{—}\langle ** \rangle$$

By $\langle * \rangle,\langle ** \rangle$ since, $M_{T(X)}(t)=M_Y(t)$ so, $T(X)=X_1+\cdots+X_n\sim \text{Poisson}(n\lambda)$

+1 (4) [+1] Show that $T(\mathbf{X})$ is a sufficient statistics for λ

$$P(X=x\mid T(X)=t)=\frac{P(X=x,T(x)=t)}{P(T(X)=t)}=\begin{cases}\dfrac{\dfrac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!}}{\dfrac{e^{-n\lambda}(n\lambda)^t}{t!}} & \text{if } \sum_{i=1}^{n}x_i=t\\ 0 & \text{if o.w.}\end{cases}$$

$$=\begin{cases}\dfrac{t!}{\prod_{i=1}^{n}x_i!\cdot n^t} & \text{if } \sum_{i=1}^{n}x_i=t\\ 0 & \text{if o.w.}\end{cases}$$

since $P(X=x\mid T(X)=t)$ does not dependent on λ

so $T(X)$ is a sufficient statistics for λ

+1 (5) [+1] Prove $\sum_{\mathbf{x}:T(\mathbf{x})=t} f_{\mathbf{X}|T}(\mathbf{x}\mid t)=1$.

$$\sum_{x:T(X)=t} f_{X|T}(x\mid t)=\sum_{x:\sum_{i=1}^{n}x_i=t}\frac{t!}{x_1!x_2!\cdots x_n!}\left(\frac{1}{n}\right)^{x_1}\left(\frac{1}{n}\right)^{x_2}\cdots\left(\frac{1}{n}\right)^{x_n}$$

since it is a sam of pmf from multinomial distribution

so $\sum_{X:T(X)=t} f_{X|T}(x\mid t)=1$