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Midterm exam, Mathematical Statistics, 2017 Fall [+ 30 points], Q1-Q5

Name: +21

+5 **Q1[+5]**. Let $X_1,\dots,X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$, $\mathbf{X}=(X_1,\dots,X_n)$, and $T(\mathbf{X})=X_1+\cdots+X_n$.

1) [+1] Derive the moment generating (mgf) function of $T(\mathbf{X})$.

+1

$M_X(t)=E(e^{tX})=\sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty}\frac{(\lambda e^t)^x e^{-\lambda}}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$

$M_{T(X)}(t)=E(e^{t(X_1+\cdots+X_n)}) = E(e^{tX_1}\cdot e^{tX_2}\cdots e^{tX_n}) \underset{\text{because iid}}{=} E(e^{tX_1})\cdots E(e^{tX_n}) = e^{\lambda n(e^t-1)}$ ✓

2) [+1] Derive the conditional pmf $f_{\mathbf{X}|T}(\mathbf{x}\,|\,t)$.

+1

$T(X) \sim \text{Poisson}(n\lambda)$

$$f_{X|T}(x|t)=\frac{P(X=x,\ \sum_{i=1}^n X_i=t)}{P(\sum_{i=1}^n X_i=t)} = \begin{cases} \dfrac{\dfrac{e^{-n\lambda}\lambda^{\sum x_i = t}}{\prod_{i=1}^n x_i!}}{\dfrac{e^{-n\lambda}(\lambda n)^t}{t!}} ✓ = \dfrac{t!}{\prod_{i=1}^n x_i!}\left(\dfrac{1}{n}\right)^t & \text{if } \sum_{i=1}^n x_i = t \\ 0 & \text{if } \sum_{i=1}^n x_i \neq t \end{cases}$$

3) [+1] Show that $T(\mathbf{X})$ is sufficient and complete for λ.

+1 By 2), because $f_{X|T}(x|t)$ doesn't depend on λ, $T(X)$ is sufficient statistic for λ

$f_T(t)=e^{-n\lambda}\frac{(n\lambda)^t}{t!}$ $t=0,1,2,\dots$

Let $g(t)$ s.t. $E_\lambda(g(t))=0$ $\forall \lambda>0$ ✓

$E(g(t))=\sum_{t=0}^{\infty} g(t)\frac{e^{-n\lambda}(n\lambda)^t}{t!}$ $\forall \lambda>0$

Write clearly. $=g(t)\sum_{t=0}^{\infty}\frac{(n\lambda)^t}{t!} = g(t)\cdot e^{nt} = 0$ $\forall \lambda>0$ $\Leftrightarrow g(t)=0$

$\therefore T(X)$ is sufficient and complete for λ

4) [+1] Derive a Bayes estimator for λ under the prior $\pi(\lambda)=\dfrac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}\exp\left(-\dfrac{\lambda}{\beta}\right)$.

+1

$f(x|\lambda)=\prod_{i=1}^n\frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$

$\pi(\lambda|x) \propto f(x|\lambda)\pi(\lambda)$

$\propto e^{-n\lambda}\lambda^{\sum_{i=1}^n x_i}\lambda^{\alpha-1}e^{-\frac{\lambda}{\beta}}$

$\propto \lambda^{\sum_{i=1}^n x_i+\alpha-1}e^{-\lambda(n+\frac{1}{\beta})}$

$\lambda|x \sim \Gamma\left(\sum_{i=1}^n x_i+\alpha,\ \frac{1}{n+\frac{1}{\beta}}\right)$

$\hat\lambda_B = E[\lambda|x] = \dfrac{\sum_{i=1}^n x_i+\alpha}{n+\frac{1}{\beta}} = \dfrac{\sum_{i=1}^n x_i\beta+\alpha\beta}{n\beta+1}$ ✓

+1 5) [+1] Express the Bayes estimator as a linear combination of the MLE and the prior mean.

$L(\lambda|x)=\dfrac{e^{-n\lambda}\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$

$\log L(\lambda|x) = -n\lambda+\sum_{i=1}^n x_i\ln\lambda-\ln\prod_{i=1}^n x_i!$

$\frac{\partial}{\partial\lambda}\log L(\lambda|x) = -n+\dfrac{\sum_{i=1}^n x_i}{\lambda} \overset{set}{=} 0$

$\hat\lambda=\bar X$

$\frac{\partial^2}{\partial\lambda^2}\log L(\lambda|x) = -\dfrac{\sum_{i=1}^n x_i}{\lambda^2}<0$ $\forall\lambda$

$\therefore \hat\lambda=\bar X$ is the MLE of λ

prior mean $=E(\lambda)=\alpha\beta$

$\hat\lambda_B = \dfrac{\sum_{i=1}^n x_i\beta+\alpha\beta}{n\beta+1} = \dfrac{n\beta}{n\beta+1}\bar X + \dfrac{1}{n\beta+1}\alpha\beta$ ✓

$\therefore \hat\lambda_B$ is a linear combination of the MLE and the prior mean

+4

$X \to Exp(-\eta)$ $\quad X_1 + \dots + X_n \sim \Gamma(n, -\eta)$

Q2 [+5] Let $X_1, \dots, X_n \overset{iid}{\sim} f(x \mid \eta) = -\eta \exp(\eta x)$, $\eta < 0$, $x > 0$.

1) [+1] Derive the MLE $\hat{\eta}$ of η.

+1

$L(\eta \mid x) = \prod_{i=1}^{n} -\eta \exp(\eta x_i) = (-\eta)^n \exp(\eta \sum_{i=1}^{n} x_i)$

$\log L(\eta \mid x) = n \ln(-\eta) + \eta \sum_{i=1}^{n} x_i$

$\frac{d}{d\eta} \log L(\eta \mid x) = \frac{+n}{+\eta} + \sum_{i=1}^{n} x_i \overset{set}{=} 0$

$-\sum_{i=1}^{n} x_i = \frac{n}{\eta}$ $\quad \eta = -\frac{1}{\bar{X}}$

$\frac{d^2}{d\eta^2} \log L(\eta \mid x) = -\frac{n}{\eta^2} < 0 \ \forall \eta$.

$\therefore \hat{\eta} = \frac{1}{\bar{X}}$ since $\begin{cases} \frac{d}{d\eta} \log L(\eta \mid x) \big|_{\hat{\eta}} = 0 \\ \frac{d^2}{d\eta^2} \log L(\eta \mid x) < 0 \ \forall \eta \end{cases}$

2) [+2] Calculate the bias $E_\eta[\hat{\eta}] - \eta$.

+2

$X \to Exp(-\eta)$ $\quad \eta < 0$

$X_1 + \dots + X_n \to \Gamma(n, -\eta)$ $\quad \eta < 0$

$f_T(t) = \frac{(-\eta)^n}{\Gamma(n)} t^{n-1} e^{\eta t}$ $\quad t > 0, \eta < 0$

$E(\frac{1}{T}) = \frac{(-\eta)^n}{\Gamma(n)} \int_0^\infty t^{n-2} e^{-(-\eta)t} \, dt$

$= \frac{(-\eta)^n}{\Gamma(n)} \cdot \frac{\Gamma(n-1)}{(-\eta)^{n-1}} = \frac{-\eta}{(n-1)}$

$E(\hat{\eta}) = E(\frac{n \eta}{n-1})$

$Bias = \frac{n\eta}{n-1} - \eta = \frac{\eta}{n-1}$

3) [+1] Show that $\frac{X_n}{X_1 + \dots + X_n}$ is an ancillary statistc for η.

$\frac{X_n}{X_1 + \dots + X_n} = \frac{\theta X_n}{\theta X_1 + \dots + \theta X_n} = \frac{Y_1}{Y_1 + \dots + Y_n}$ $\quad$ Let $Y = X\eta$ $\quad \eta < 0$

$X = \frac{Y}{\eta}$ $\quad \left|\frac{dx}{dy}\right| = \left|\frac{1}{\eta}\right|$

Use η

$f_Y(y) = (-\eta) \cdot \frac{1}{\eta} e^{y} = -e^{y}$ $\quad y > 0$ doesn't depend on θ

$\therefore Y$ doesn't depend on θ, and $\frac{Y_1}{Y_1 + \dots + Y_n}$ also doesn't depend on θ

$\therefore \frac{X_n}{X_1 + \dots + X_n}$ is an ancillary statistic for η.

4) [+1] Calculate $E_\eta\left[\frac{X_n}{X_1 + \dots + X_n}\right]$ (need proofs).

+1

$f(x \mid \eta) = -\eta \exp(\eta x)$

$f(x \mid \eta) = h(x) c(\eta) \exp\left[\sum_{i=1}^{n} w_i(x) t_i(\eta)\right]$

and $c(\eta) = \eta$ $\quad w(x) = x$ $\quad t(\eta) = \eta$

$f(x \mid \eta)$ is one dimension natural exponentinal family

and $t(\eta) = \eta < 0$ has one open set

$\therefore \sum_{i=1}^{n} X_i$ is complete statistic for η.

By basu theorem ancillary statistic $\perp\!\!\!\perp$ c.s.s.

$E(X_n) = E\left(\frac{X_n}{X_1 + \dots + X_n} \cdot (X_1 + \dots + X_n)\right)$

$= E\left[\frac{X_n}{X_1 + \dots + X_n}\right] E\left[X_1 + \dots + X_n\right]$

$E(X_n) = \frac{1}{\eta}$. $\quad E(X_1 + \dots + X_n) = \frac{-n}{\eta}$

$\therefore E\left[\frac{X_n}{X_1 + \dots + X_n}\right] = \frac{\frac{1}{\eta}}{\frac{-n}{\eta}} = \frac{1}{n}$

Q3 [+5] +3

(1) [+1] State Basu's theorem [+1]

+1

If $C(X)$ is a complete sufficient statistic for θ and $T(X)$ is a ancillary statistic for θ then $C(X) \perp\!\!\!\perp T(X)$ ✓

(2) [+2] Prove Basu's theorem [+2]

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CRLB

(3) [+2] Let $T(\mathbf{X})$ be a statistic, where $\mathbf{X}$ has the pdf $f_{\mathbf{X}}(\mathbf{x}\,|\,\theta)$. Assume that the pdf allows you to do some interchange of integral and differentiation. Derive the Cramér–Rao lower bound.

+2

$$Var_\theta(T(\mathbf{X})) \geq \frac{\left(\frac{d}{d\theta} E[T(X)]\right)^2}{E\left[\frac{d}{d\theta} \ln f(x|\theta)\right]^2}$$

Proof

$$\frac{d}{d\theta} E[T(X)] = \frac{d}{d\theta} \int T(X) f(x|\theta)\,dx$$
$$= \int T(X) \frac{d}{d\theta} f(x|\theta)\,dx$$
$$= \int T(X) \frac{d}{d\theta} \ln f(x|\theta) f(x|\theta)\,dx$$
$$= E\left[T(X) \frac{d}{d\theta} \ln f(x|\theta)\right]$$
$$= \mathrm{cov}\left[T(X) \frac{d}{d\theta} \ln f(x|\theta)\right]$$

$$\because E\left(\frac{d}{d\theta} \ln f(x|\theta)\right)$$
$$= \int \frac{d}{d\theta} \ln f(x|\theta) f(x|\theta)\,dx$$
$$= \int \frac{\frac{d}{d\theta} f(x|\theta)}{f(x|\theta)} f(x|\theta)\,dx = \int 1\,dx = 0$$

$$\left(\frac{d}{d\theta} E[T(X)]\right)^2 = \mathrm{cov}^2\left[T(X) \frac{d}{d\theta} \ln f(x|\theta)\right]$$

by cauchy schwarz inequality

$$\leq \mathrm{Var}(T(X))\,\mathrm{Var}\left(\frac{d}{d\theta} \ln f(x|\theta)\right)$$
$$= \mathrm{Var}(T(X))\,E\left[\frac{d}{d\theta} \ln f(x|\theta)\right]^2$$

$$\therefore \mathrm{Var}(T(X)) \geq \frac{\left(\frac{d}{d\theta} E[T(X)]\right)^2}{E\left[\frac{d}{d\theta} \ln f(x|\theta)\right]^2}$$ ✓

+1

Q4 [+5] Let $X_1,\dots,X_n \overset{iid}{\sim} f(x\mid\theta)=2\theta^2x^{-3}I(x\ge\theta)$, $\theta>0$.

+1 (1) [+2] Derive a size-α LR test for testing $H_0:\theta\le\theta_0$ vs. $H_1:\theta>\theta_0$.

$\Theta_0=\{\theta\le\theta_0\}$ $\Theta=\{\theta>0\}$

$L(\theta;\mathbf{x})=\prod_{i=1}^n 2\theta^2 x_i^{-3} I(x_i\ge\theta)=2^n\theta^{2n}\prod_{i=1}^n x_i^{-3} I_{(0,x_{(1)}]}(\theta)$

$\because L(\theta;\mathbf{x})$ is increasing in θ, when $\theta\in(0,x_{(1)})$

$\therefore \hat\theta = X_{(1)}$, $\hat\theta_0=\begin{cases}X_{(1)} & \text{if } X_{(1)}\ge\theta_0\\ \theta_0 & \text{if } X_{(1)}<\theta_0\end{cases}$

×

$L(\hat\theta_0)=\begin{cases}2^n(X_{(1)})^{2n}\prod_{i=1}^n x_i^{-3}, & X_{(1)}\ge\theta_0\\ 2^n(\theta_0)^{2n}\prod_{i=1}^n x_i^{-3}, & X_{(1)}<\theta_0\end{cases}$

?

$\lambda(\mathbf{x})=\frac{L(\hat\theta_0)}{L(\hat\theta)}=\begin{cases}1, & X_{(1)}\ge\theta_0\\ \left[\frac{\theta_0}{X_{(1)}}\right]^{2n}, & X_{(1)}<\theta_0\end{cases}$

$\lambda(\mathbf{x})\le c \iff \left[\frac{\theta_0}{X_{(1)}}\right]^{2n}\le c$ & $X_{(1)}\le\theta_0$ for some $0<c<1$

$\iff X_{(1)}>c^*$ for some c^*

and c^* satisfy $P_{\theta_0}(X_{(1)}>c^*)=\alpha$

$\int_{c^*}^{\infty}2\theta_0^2x^{-3}dx=\frac{2\theta_0^2}{(-2)}x^{-2}\Big|_{c^*}^{\infty}=\theta_0^2c^{*(-2)}=\alpha$

$(c^*)^{-2}=\frac{\alpha}{\theta_0^2}$ $c^*=\theta_0\alpha^{-\frac12}$

$R=\{\mathbf{x}: X_{(1)}>\theta_0\alpha^{-\frac12}\}$ is the LR test of size α

+0 (2) [+1] Derive the power function.

$\beta(\theta)=P_\theta\left(X_{(1)}>\left(\frac{\alpha}{\theta_0}\right)^{-\frac12}\right)$

$=\left(P_\theta(X>\theta_0\alpha^{-\frac12})\right)^n=\left(\int_{\theta_0\alpha^{-\frac12}}^{\infty}2\theta^2x^{-3}dx\right)^n$

$=\left(\left(\frac{\theta}{\theta_0}\right)^2\alpha\right)^n=\left(\frac{\theta}{\theta_0}\right)^{2n}\alpha^n$ ×

+0 (3) [+1] Draw figures of the power function under $\theta_0=1$, $\alpha=0.5$, $n=1$ and 2 (details).

$n=1$ $\beta(\theta)=\left(\frac{\theta}{\theta_0}\right)^2\alpha$

×

$\beta'(\theta)=2\left(\frac{\theta}{\theta_0}\right)\cdot\frac{1}{\theta_0}\alpha$

$\beta''(\theta)=\frac{2}{\theta_0^2}\alpha$

+0 (4) [+1] Derive a $(1-\alpha)$ one-sided CI by inverting a test $H_0:\theta=\theta_0$ vs. $H_1:\theta>\theta_0$.

$R(\theta_0)=\{\mathbf{x}\mid X_{(1)}>\theta_0\alpha^{-\frac12}\}$

$A(\theta_0)=\{\mathbf{x}\mid X_{(1)}<\theta_0\alpha^{-\frac12}\}$

$C(\mathbf{x})=\{\theta\mid X_{(1)}<\theta\alpha^{-\frac12}\}$

θ $(1-\alpha)\%$ C.I $=\left[\frac{X_{(1)}}{\alpha^{-\frac12}},\infty\right)$

×

+7 +8

Prove it

Q5 [+10] Let $X_1,\ldots,X_n \overset{iid}{\sim} N(\theta,\sigma^2)$, where σ^2 is known.

-1

+1 (1) [+2] Derive a level-α UMP test for $H_0:\theta\le\theta_0$ vs. $H_1:\theta>\theta_0$ (with proof)

$f(x|\theta)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{1}{2\sigma^2}(x-\theta)^2\right]$ is a one-dimensional exponential family

$=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2\right]$? $f(x|\theta)$ is increasing in $\bar{x}$, $\therefore \bar{X}$ has mlr property in θ

By Karlin Rubin $R=\{x \mid \bar{X}>c\}$ is LR test

Explain what is this

and c determined by

$P_{\theta_0}(\bar{X}>c)=P\left(\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}}>\frac{c-\theta_0}{\sigma/\sqrt{n}}\right)=\alpha$

$Z_\alpha=\frac{c-\theta_0}{\sigma/\sqrt{n}}$ $c=\theta_0+Z_\alpha\frac{\sigma}{\sqrt{n}}$

$\therefore R=\{x \mid \bar{X}>\theta_0+Z_\alpha\frac{\sigma}{\sqrt{n}}\}$

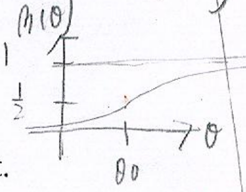
+2 (2) [+2] Is this test unbiased? Prove or disprove your answer.

$\beta(\theta)=P_\theta\left(\bar{X}>\theta_0+Z_\alpha\frac{\sigma}{\sqrt{n}}\right)=P\left(\frac{\bar{X}-\theta}{\sigma/\sqrt{n}}>\frac{\theta_0-\theta}{\sigma/\sqrt{n}}+Z_\alpha\right)=1-\Phi\left(\frac{\theta_0-\theta}{\sigma/\sqrt{n}}+Z_\alpha\right)$

$\beta'(\theta)=\frac{1}{\sigma/\sqrt{n}}\phi\left(\frac{\theta_0-\theta}{\sigma/\sqrt{n}}+Z_\alpha\right)$

$\beta''(\theta)=-\left(\frac{1}{\sigma/\sqrt{n}}\right)^2\phi'\left(\frac{\theta_0-\theta}{\sigma/\sqrt{n}}+Z_\alpha\right)$

	$-\infty$, $\theta<\theta_0$	$\theta=\theta_0$	$\theta>\theta_0$ ∞
$\beta(\theta)$	0	$\frac{1}{2}$	1
$\beta'(\theta)$	+	+	+
$\beta''(\theta)$		0	



$\beta(\theta'')>\beta(\theta')$

$\forall\ \theta''\in\Theta_1,\ \theta'\in\Theta_0$

$\therefore$ is unbiased.

+1 (3) [+1] Derive a P-value for the above UMP test.

$W(X)=\bar{X}$

$\text{p-value}=P(W(X)>W(x))$

$=\sup_{\theta\in\Theta_0}P\left(\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}}>\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}}\right)$

$=1-\Phi\left(\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}}\right)$

+1 ~~+0~~ (4) [+1] Derive a level-α unbiased test for $H_0:\theta=\theta_0$ vs. $H_1:\theta\ne\theta_0$ from the LR test.

$\log L(\theta;x)=-\frac{n}{2}\ln(2\pi\sigma^2)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2$

$\frac{\partial}{\partial\theta}\log L(\theta;x)=\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2\overset{\text{set}}{=}0$ $\hat{\theta}=\bar{X}$

$\hat{\theta}_0=\begin{cases}\bar{X} & \text{if } \bar{X}<\theta_0\\ \theta_0 & \text{if } \bar{X}>\theta_0\end{cases}$

$L(\hat{\theta}_0)=\begin{cases}(2\pi\sigma^2)^{-\frac{n}{2}} & \bar{X}<\theta_0\\ (2\pi\sigma^2)^{-\frac{n}{2}}\exp\left[-\frac{n}{2\sigma^2}(\bar{X}-\theta_0)^2\right] & \bar{X}>\theta_0\end{cases}$

$\lambda(x)=\begin{cases}1 & \bar{X}<\theta_0\\ \exp\left[-\frac{n}{2\sigma^2}(\bar{X}-\theta_0)^2\right] & \bar{X}>\theta_0\end{cases}$

$\lambda(x)<c \iff \exp\left[-\frac{n}{2\sigma^2}(\bar{X}-\theta_0)^2\right]<c$ and $\bar{X}>\theta_0$

$\iff |\bar{X}-\theta_0|>c^*$

and c^* determined by $P_{\theta_0}(|\bar{X}-\theta_0|>c^*)=\alpha$

$P_{\theta_0}(\bar{X}-\theta_0>c^*,\ \bar{X}-\theta_0<-c^*)$

$=2P\left(\left|\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}}\right|>\frac{c^*}{\sigma/\sqrt{n}}\right)=\alpha$

$c^*=\frac{\sigma}{\sqrt{n}}Z_{\frac{\alpha}{2}}$, $R=\{x \mid |\bar{X}-\theta_0|>\frac{\sigma}{\sqrt{n}}Z_{\frac{\alpha}{2}}\}$

+1 (5) [+1] Derive a P-value for the above test ($H_0:\theta=\theta_0$ vs. $H_1:\theta\ne\theta_0$)

Let $W(X)=|\bar{X}-\theta_0|$

$\text{p-value}=P(|\bar{X}-\theta_0|>|\bar{x}-\theta_0|)$

$=P\left(\frac{|\bar{X}-\theta_0|}{\sigma/\sqrt{n}}>\frac{|\bar{x}-\theta_0|}{\sigma/\sqrt{n}}\right)$

$=2\left[1-\Phi\left(\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}}\right)\right]$

+1 (6) [+2] Derive the power function of the above test ($H_0:\theta=\theta_0$ vs. $H_1:\theta\ne\theta_0$) and draw its figure.

$\beta_0=P\left(|\bar{X}-\theta_0|>\frac{\sigma}{\sqrt{n}}Z_{\frac{\alpha}{2}}\right)$

$=P\left(\bar{X}>\theta_0+\frac{\sigma}{\sqrt{n}}Z_{\frac{\alpha}{2}},\ \bar{X}<\theta_0-\frac{\sigma}{\sqrt{n}}Z_{\frac{\alpha}{2}}\right)$

$=P\left(\frac{\bar{X}-\theta}{\sigma/\sqrt{n}}>\frac{\theta_0-\theta}{\sigma/\sqrt{n}}+Z_{\frac{\alpha}{2}},\ \frac{\bar{X}-\theta}{\sigma/\sqrt{n}}<\frac{-(\theta_0-\theta)}{\sigma/\sqrt{n}}Z_{\frac{\alpha}{2}}\right)$

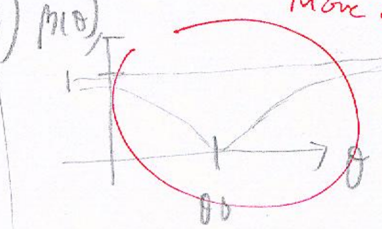
$=1-2\Phi\left(\frac{\theta_0-\theta}{\sigma/\sqrt{n}}Z_{\frac{\alpha}{2}}\right)$

$\beta'(\theta)=\frac{2}{\sigma/\sqrt{n}}\phi\left(\frac{\theta_0-\theta}{\sigma/\sqrt{n}}Z_{\frac{\alpha}{2}}\right)$

$\beta''(\theta)=\frac{-2^2}{(\sigma/\sqrt{n})^2}\phi'\left(\frac{\theta_0-\theta}{\sigma/\sqrt{n}}Z_{\frac{\alpha}{2}}\right)$

	$-\infty$ $\theta<\theta_0$	$\theta=\theta_0$	$\theta>\theta_0$ ∞
$\beta(\theta)$	1	0	1
$\beta'(\theta)$		0	
$\beta''(\theta)$		0	

Write more clearly



+1 (7) [+1] Use a pivot to derive a (1-α) CI for θ when σ^2 is unknown.

under Ho $\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}}\to N(0,1)$

$\frac{(n-1)S^2}{\sigma^2}\to\chi^2(n-1)$ $A(\theta)=P\left(\frac{|\bar{X}-\theta|}{S/\sqrt{n}}\le t_{\frac{\alpha}{2}}(n-1)\right)$

$\frac{\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2(n-1)}}}=\frac{\bar{X}-\theta_0}{S/\sqrt{n}}\to t(n-1)$ $=P\left(\bar{X}-\frac{S}{\sqrt{n}}t_{\frac{\alpha}{2}}(n-1)\le\theta\le\bar{X}+\frac{S}{\sqrt{n}}t_{\frac{\alpha}{2}}(n-1)\right)$

$\therefore$ $(1-\alpha)$ C.I for θ is

$\left[\bar{X}-\frac{S}{\sqrt{n}}t_{\frac{\alpha}{2}}(n-1),\ \bar{X}+\frac{S}{\sqrt{n}}t_{\frac{\alpha}{2}}(n-1)\right]$