Homework#7 Statistical Inference
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Derive the MLE ($\hat{\mu}$) under $\mathrm{DE}(\mu,\lambda)$.

Solution:

$$X_1,\dots,X_n \text{ be iid } \mathrm{DE}(\mu,\lambda), \mu \in \mathbb{R}, \lambda > 0$$

$$\mathrm{f}(\mathrm{x};\mu,\lambda) = \frac{1}{2\lambda}\exp(-\frac{|x-\mu|}{\lambda})$$

$$\mathrm{L}(\mu,\lambda) = \prod_{i=1}^{n} \mathrm{f}(\mathrm{x}_i;\mu,\lambda) = \prod_{i=1}^{n} \frac{1}{2\lambda}\exp\left(-\frac{|x_i-\mu|}{\lambda}\right) = (2\lambda)^{-n}\exp\left(-\frac{\sum_{i=1}^{n}|x_i-\mu|}{\lambda}\right)$$

$$\ln \mathrm{L}(\mu,\lambda) = -n\ln(2\lambda) - \frac{\sum_{i=1}^{n}|x_i-\mu|}{\lambda}$$

Fixed λ, then $\max_\mu\{\ln \mathrm{L}(\mu,\lambda)\} \Leftrightarrow \min_\mu \sum_{i=1}^{n}|x_i-\mu|$

Let $g(\mu) = \sum_{i=1}^{n}|x_i-\mu| = \sum_{i=1}^{n}[(x_i-\mu)I(x_i>\mu) + (\mu-x_i)I(x_i<\mu)]$

$$\Rightarrow \frac{d}{d\mu}g(\mu) = \sum_{i=1}^{n}[(-1)I(x_i>\mu) + (1)I(x_i<\mu)]$$

$$= (-1)\sum_{i=1}^{n} I(x_i>\mu) + \sum_{i=1}^{n} I(x_i<\mu) = 0\ (set)$$

$$\Rightarrow \sum_{i=1}^{n} I(x_i>\mu) = \sum_{i=1}^{n} I(x_i<\mu) \Rightarrow \hat{\mu} = M_n$$

Where, M_n is median of $X_1,\dots,X_n$.

Case1: $\mathrm{n} = 2\mathrm{m}+1, \forall\ \mathrm{m} \in \mathbb{N}$

$$\because \frac{d}{d\mu}g(\mu) = \begin{cases} positive\ if\ \mu \geq X_{\left(\frac{n+1}{2}+1\right)} \\ negative\ if\ \mu \leq X_{\left(\frac{n+1}{2}-1\right)} \\ 0\ if\ \mu = X_{\left(\frac{n+1}{2}\right)} = M_n \end{cases}$$

$$\therefore g(\hat{\mu})\ is\ \min_{\mu}\sum_{i=1}^{n}|x_i-\mu|$$

Case2: $\mathrm{n} = 2\mathrm{m}, \forall\ \mathrm{m} \in \mathbb{N}$

$$\because \frac{d}{d\mu} g(\mu) = \begin{cases} positive \ if \ \mu \geq X_{\left(\frac{n}{2}+1\right)} \\ negative \ if \ \mu \leq X_{\left(\frac{n}{2}\right)} \\ 0 \ if \ \mu = \dfrac{X_{\left(\frac{n}{2}+1\right)} + X_{\left(\frac{n}{2}\right)}}{2} = M_n \end{cases}$$

$$\therefore \mathrm{g}(\hat{\mu}) \ is \ \min_{\mu} \sum_{i=1}^{n} |x_i - \mu|$$

Take $\mu = \hat{\mu}$ in to $\ln \mathrm{L}(\mu, \lambda)$

$$\ln \mathrm{L}(\lambda) = -n \ln(2\lambda) - \frac{\sum_{i=1}^{n} |x_i - \hat{\mu}|}{\lambda}$$

Let $\mathrm{k} = \sum_{i=1}^{n} |x_i - \hat{\mu}|$

$$\ln \mathrm{L}(\lambda) = -n \ln(2\lambda) - \frac{k}{\lambda}$$

$$\Longrightarrow \frac{d}{d\lambda} \ln \mathrm{L}(\lambda) = \frac{-n}{\lambda} + \frac{k}{\lambda^2} = 0 \ (\text{set}) \Longrightarrow \hat{\lambda} = \frac{k}{n} = \frac{\sum_{i=1}^{n} |x_i - \hat{\mu}|}{n}$$

$$\frac{d^2}{d\lambda^2} \ln \mathrm{L}(\lambda) \bigg|_{\lambda = \hat{\lambda}} = \frac{n}{\lambda^2} - \frac{2k}{\lambda^3} \bigg|_{\lambda = \hat{\lambda}} = \frac{n^3}{k^2} - \frac{2n^3 k}{k^3} = -\frac{n^3}{k^2} < 0$$

$$\text{The MLE of } (\mu, \lambda) \text{is} \begin{cases} \hat{\mu} = \mathrm{M_n} \\ \hat{\lambda} = \dfrac{\sum_{\mathrm{i}=1}^{\mathrm{n}} |\mathrm{x_i} - \hat{\mu}|}{\mathrm{n}} \end{cases}$$