HW#5

Q2 [+5] Let $X_1, \ldots, X_n \overset{iid}{\sim} f(x|\eta) = -\eta \exp(\eta x)$, $\eta < 0$, $x > 0$.

(1) [+1] Derive the MLE $\hat{\eta}$ of η.

Solution 1).

The likelihood and log-likelihood functions are

$$L(\eta) = \prod_{i=1}^{n} f(X_i|\eta) = (-\eta)^n \exp\left(\eta \sum_{i=1}^{n} X_i\right) \quad \text{and} \quad \ell(\eta) = \log L(\eta) = n\log(-\eta)^n + \eta \sum_{i=1}^{n} X_i .$$

Thus, the MLE of η is the solution of the score function $\partial \ell(\eta)/\partial \eta = 0$ which is $\hat{\eta} = -n/\Sigma_{i=1}^{n} X_i$.

The MLE attains the maximum of the log-likelihood value since $\partial^2 \ell(\eta)/\partial \eta^2 |_{\eta=\hat{\eta}} = -n/\hat{\eta}^2 < 0$.

(2) [+2] Calculate the bias $E_\eta[\hat{\eta}] - \eta$.

Solution (2).

Since $X_i \sim f(x|\eta) = -\eta \exp(\eta x)$, $i = 1, 2, \ldots, n$, we have

$$\sum_{i=1}^{n} X_i \equiv T \sim f(t|\eta) = \frac{(-\eta)^n}{\Gamma(n)} t^{n-1} \exp(\eta t), \qquad \eta < 0,\ t > 0 .$$

Therefore,

$$E(\hat{\eta}) = -nE(T^{-1}) = \int_0^\infty t^{-1} \frac{(-\eta)^n}{\Gamma(n)} t^{n-1} \exp(\eta t) dt = \frac{n\eta}{n-1} .$$

Then the bias is

$$E(\hat{\eta}) - \eta = \frac{\eta}{n-1} .$$

(3) [+1] Show that $\dfrac{X_n}{X_1 + \cdots + X_n}$ is an ancillary statistic for η.

Solution (3).

We have $X_n/\Sigma_{i=1}^{n} X_i = (-\eta X_n)/(-\eta \Sigma_{i=1}^{n} X_i)$. Since $-\eta X_n \sim \text{Exp}(1)$ and $-\eta \Sigma_{i=1}^{n} X_i \sim \Gamma(n,1)$, the distribution of $X_n/\Sigma_{i=1}^{n} X_i$ does not depend on parameter η. Then we have shown that $X_n/\Sigma_{i=1}^{n} X_i$ is an ancillary statistic for η.

(4) [+1] Calculate $E_\eta\left[\dfrac{X_n}{X_1+\cdots+X_n}\right]$ (need proofs).

Solution (4).

We have

$$f(x_1,\ldots,x_n\mid\eta)=\prod_{i=1}^{n} f(x_i\mid\eta)=(-\eta)^n\exp\left(\eta\sum_{i=1}^{n}x_i\right)=\exp\left(\eta\sum_{i=1}^{n}x_i+n\log(-\eta)\right).$$

Thus, the canonical form of this one-dimensional exponential family is

$$f(x_1,\ldots,x_n\mid\eta)=\exp\{\eta T(x_1,\ldots,x_n)-A(\eta)\}h(x_1,\ldots,x_n),$$

where $T(x_1,\ldots,x_n)=\Sigma_{i=1}^n x_i$, $A(\eta)=-n\log(-\eta)$, and $h(x_1,\ldots,x_n)=1$. Since the natural parameter space $\Theta=\{\eta\mid\eta<0\}$ contains an one-dimensional rectangle (e.g., $(-2,-1)$), we obtain that $T(x_1,\ldots,x_n)=\Sigma_{i=1}^n x_i$ is complete.

Now, by Basu's Theorem, $\Sigma_{i=1}^n x_i$ (complete sufficient statistic) and $X_n/\Sigma_{i=1}^n X_i$ (ancillary statistic) are independent. Therefore, we have

$$E_\eta(X_n)=E_\eta\left(\frac{X_n}{\Sigma_{i=1}^n X_i}\cdot\Sigma_{i=1}^n X_i\right)=E_\eta\left(\frac{X_n}{\Sigma_{i=1}^n X_i}\right)E_\eta(\Sigma_{i=1}^n X_i).$$

This implies

$$E_\eta\left(\frac{X_n}{\Sigma_{i=1}^n X_i}\right)=\frac{E_\eta(X_n)}{E_\eta(\Sigma_{i=1}^n X_i)}=\frac{-1/\eta}{-n/\eta}=\frac{1}{n}.$$

NOTE: One may also apply the change of variables to show that $X_n/\Sigma_{i=1}^n X_i$ follows a Beta distribution which does not depend on η and then compute its expectation.

Q4 [+5] Let $X_1, \ldots, X_n \overset{iid}{\sim} f(x|\theta) = 2\theta^2 x^{-3} I(x \ge \theta)$, $\theta > 0$.

(1) [+2] Derive a size-α LR test for testing $H_0: \theta \le \theta_0$ vs. $H_1: \theta > \theta_0$.

Solution (1).

The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(X_i|\theta) = 2^n \theta^{2n} \prod_{i=1}^{n} X_i^{-3} I(X_{(1)} \ge \theta),$$

where $X_{(1)} = \min(X_1, \ldots, X_n)$. Clearly, the likelihood function $L(\theta)$ is increasing in θ.

If $X_{(1)} \le \theta_0$, we have the likelihood ratio (LR) test statistic $\lambda = 1$ which implies that we always accept the null hypothesis. This is a natural result since $\theta \le X_{(1)} \le \theta_0$. If $X_{(1)} > \theta_0$, the LR test statistic is

$$\lambda = \frac{\sup_{\theta \in \Theta_0} L(\theta)}{\sup_{\theta \in \Theta} L(\theta)} = \frac{2^n \theta_0^{2n} \prod_{i=1}^{n} X_i^{-3} I(X_{(1)} \ge \theta_0)}{2^n X_{(1)}^{2n} \prod_{i=1}^{n} X_i^{-3} I(X_{(1)} \ge X_{(1)})} = \left(\frac{\theta_0}{X_{(1)}} \right)^{2n},$$

where $\Theta_0 = \{\theta | \theta \le \theta_0\}$ and $\Theta_1 = \{\theta | \theta > \theta_0\}$, and $\Theta = \Theta_0 \cup \Theta_1 = \{\theta | \theta \in \mathrm{R}\}$.

Since $(\theta_0 / X_{(1)})^{2n} < c$ for some c is equivalent to $X_{(1)} > k$ for some k. Now, we consider

$$\phi(x) = \begin{cases} 1 & \text{if } x_{(1)} > k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha = E_{\theta_0}\{\phi(X)\} = \Pr(X_{(1)} > k) = (\theta_0 / k)^{2n}$. This implies $k = \theta_0 \alpha^{-1/2n}$. Thus, the desired LR test with size α is

$$\phi(x) = \begin{cases} 1 & \text{if } x_{(1)} > \theta_0 \alpha^{-1/2n}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) [+1] Derive the power function.

Solution (2).

The power function is defined as

$$\beta(\theta) = E_\theta\{\phi(X)\} = \Pr(X_{(1)} > \theta_0 \alpha^{-1/2n}), \qquad \theta > 0.$$

If $\theta_0 \alpha^{-1/2n} \le \theta$, one has $\beta(\theta) = 1$; if $\theta_0 \alpha^{-1/2n} > \theta$, one has $\beta(\theta) = (\theta / \theta_0)^{2n} \alpha$. Therefore, the power function of the LR test in (1) is

$$\beta(\theta) = \begin{cases} (\theta/\theta_0)^{2n} \alpha & \text{if } \theta_0 \alpha^{-1/2n} > \theta, \\ 1 & \text{if } \theta_0 \alpha^{-1/2n} \le \theta. \end{cases}$$

(3) [+1] Draw figures of the power function under $\theta_0 = 1$, $\alpha = 0.5$, $n = 1, 2$ (details).

Solution (3).

Figure 1 plots the power function under $\theta_0 = 1$, $\alpha = 0.5$, $n = 1, 2$.

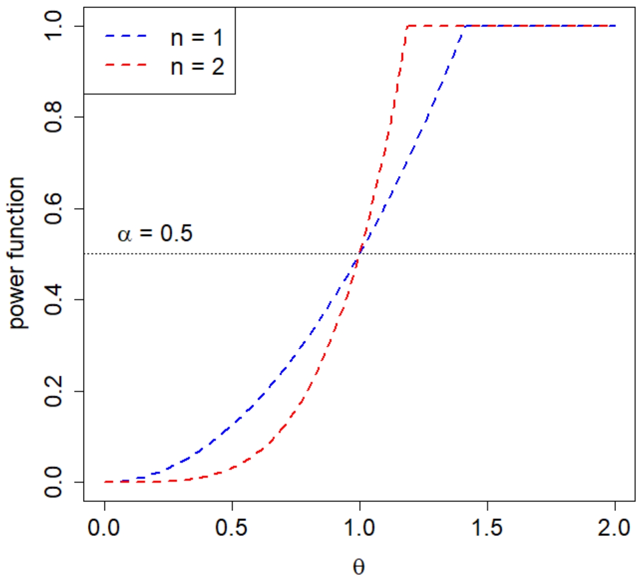

Figure 1. Power functions with sample size $n = 1, 2$ under $\theta_0 = 1$, $\alpha = 0.5$.

(4) [+1] Derive a $(1-\alpha)$ one-sided CI by inverting a test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$.

Solution (4).

Similarly, one only needs to consider the case $X_{(1)} > \theta_0$. The LR test statistic is

$$\lambda = \frac{\sup_{\theta \in \Theta_0} L(\theta)}{\sup_{\theta \in \Theta} L(\theta)} = \frac{2^n \theta_0^{2n} \prod_{i=1}^{n} X_i^{-3} I(X_{(1)} \geq \theta_0)}{2^n X_{(1)}^{2n} \prod_{i=1}^{n} X_i^{-3} I(X_{(1)} \geq X_{(1)})} = \left(\frac{\theta_0}{X_{(1)}} \right)^{2n},$$

where $\Theta_0 = \{ \theta_0 \}$ and $\Theta_1 = \{ \theta \mid \theta > \theta_0 \}$, and $\Theta = \Theta_0 \cup \Theta_1 = \{ \theta \mid \theta \geq \theta_0 \}$. Then the test is exactly the same as in (1), that is

$$\phi(x) = \begin{cases} 1 & \text{if } x_{(1)} > \theta_0 \alpha^{-1/2n}, \\ 0 & \text{if } x_{(1)} \leq \theta_0 \alpha^{-1/2n}. \end{cases}$$

By inverting its acceptance region, we obtain a $(1-\alpha)$ confidence $[\alpha^{1/2n} x_{(1)}, \infty)$.

Exercise 9.54

Solution (a).

We have $X \sim N(\mu, \sigma^2)$ and $\nu S^2/\sigma^2 \sim \chi^2_{\mathrm{df}=\nu}$ are independent. The interval estimator is $C(x) = [x - cs, x + cs]$ and the loss function is $L\{(\mu, \sigma^2), C\} = b\mathrm{Length}(C)/\sigma - I_C(\mu)$. Thus, the risk function is

$$\begin{aligned} R\{(\mu,\sigma^2),C\} &= E[L\{(\mu,\sigma^2),C\}] = bE\{\mathrm{Length}(C)\}/\sigma - \Pr(\mu \in C) \\ &= 2bcE(S/\sigma) - \Pr(X - cS \le \mu \le X + cS) \\ &= 2bcE(S/\sigma) - \Pr\left(\left| \frac{X-\mu}{S} \right| \le c \right) \\ &= 2bcE(S/\sigma) - 2\Pr\left(\frac{X-\mu}{S} \le c \right) + 1. \end{aligned}$$

Recall

$$M \equiv E(S/\sigma) = \frac{\sqrt{2}\Gamma((\nu+1)/2)}{\sqrt{\nu}\Gamma(\nu/2)} \quad \text{and} \quad T \equiv \frac{X-\mu}{S} = \frac{(X-\mu)/\sigma}{\sqrt{S^2/\sigma^2}} \sim t_{\mathrm{df}=\nu}.$$

Thus, we obtain

$$R\{(\mu,\sigma^2),C\} = 2bcM - 2\Pr(T \le c) + 1,$$

Solution (b).

The first derivative of the risk function with respect to c is

$$\frac{\partial}{\partial c} R\{(\mu,\sigma^2),C\} = 2bM - 2f_T(c),$$

where $f_T(t)$ is the density function of $t_{\mathrm{df}=\nu}$. Set the equation equals to zero and it becomes

$$\frac{2\sqrt{2}\Gamma((\nu+1)/2)}{\sqrt{\nu}\Gamma(\nu/2)} b - \frac{2\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)} \left(1 + \frac{c^2}{\nu}\right)^{-(\nu+1)/2} = 0.$$

The above equality may hold if $b \le 1/\sqrt{2\pi}$. With some further simplification, we have that c minimize the risk function satisfies

$$b = \frac{1}{\sqrt{2\pi}} \left(\frac{\nu}{\nu + c^2} \right)^{(\nu+1)/2}.$$

Solution (c).

Let $\nu \to \infty$, we have

$$b = \lim_{\nu\to\infty} \frac{1}{\sqrt{2\pi}} \left(\frac{\nu}{\nu + c^2} \right)^{(\nu+1)/2} = \frac{1}{\sqrt{2\pi}} \lim_{\nu\to\infty} \left(1 + \frac{-c^2}{\nu + c^2} \right)^{(\nu+1)/2}$$
$$= \frac{1}{\sqrt{2\pi}} \lim_{\nu\to\infty} \left(1 + \frac{-c^2/2}{(\nu + c^2)/2} \right)^{(\nu+1)/2} = \frac{1}{\sqrt{2\pi}} e^{-c^2/2}.$$

Then,

$$c = \sqrt{-2\log(\sqrt{2\pi} b)}\,.$$

Hence we have shown that as $\nu \to \infty$, the solution will converge to the known case.

Exercise 9.55

Solution.

We have

$$\begin{aligned} R\{(\mu, \sigma^2), C\} &= E[L\{(\mu, \sigma^2), C\}] \\ &= E[L\{(\mu, \sigma^2), C\} \mid S < K]\Pr(S < K) + E[L\{(\mu, \sigma^2), C\} \mid S > K]\Pr(S > K) \\ &= R\{(\mu, \sigma^2), C'\} + E[L\{(\mu, \sigma^2), C\} \mid S > K]\Pr(S > K). \end{aligned}$$

The last equality holds since $C' = \phi$ if $S > K$. Thus, it suffices to find when

$$E[L\{(\mu, \sigma^2), C\} \mid S > K] > 0\,.$$

By straightforward calculations,

$$\begin{aligned} E[L\{(\mu, \sigma^2), C\} \mid S > K] &= E\{b\text{Length}(C) - I_C(\mu) \mid S > K\} \\ &= E\{2bcS - I_C(\mu) \mid S > K\} \\ &= E\{2bcK - 1 \mid S > K\} \\ &= 2bcK - 1. \end{aligned}$$

Therefore, if $K > 1/2bc$ then C' dominates C.