#4 Homework of Mathematical Statistics (Chapter 8)
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Problem

Let $X_1, \ldots, X_n \overset{iid}{\sim} f(x|\theta) = \theta x^{-2} I(x \geq \theta)$, where $\theta > 0$. Consider testing

$H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$ for some know value θ_0.
(a) Derive the LR test with size α.
(b) Derive the power function.
(c) Using R, draw a figure of the power function under $\alpha = 0.05$ with 3 different values of n. (Combine 3 curves in one figure). Explain how the changes under different values of n.

Solution:

$X_1, \ldots, X_n \overset{iid}{\sim} f(x|\theta) = \theta x^{-2} I(x \geq \theta)\ \theta > 0$
To test $H_0 : \theta \leq \theta_0\ v.s. H_1 : \theta > \theta_0$
Ⓗ$= \{\theta | 0 < \theta < \infty\}$
Ⓗ0$= \{\theta | 0 < \theta \leq \theta_0\}$

(a)&(b)

$$\mathrm{L}(\theta|\boldsymbol{x}) = f(\boldsymbol{x}|\theta) = \prod_{i=1}^{n} \frac{\theta}{x_i^2} I(x_i \geq \theta) = \frac{\theta^n}{\prod_{i=1}^{n} {x_i}^2} I\left(x_{(1)} \geq \theta\right)$$

$\because$ $\mathrm{L}(\theta|\boldsymbol{x})$ is increasing in $\theta \in (0, x_{(1)}]$
$\therefore$ $\hat{\theta} = x_{(1)}$ is the mle of θ.

$$\Rightarrow \sup_{\theta \in Ⓗ} \mathrm{L}(\theta|\boldsymbol{x}) = \frac{[x_{(1)}]^2}{\prod_{i=1}^{n} {x_i}^2}$$

$$\sup_{\theta \in Ⓗ0} \mathrm{L}(\theta|\boldsymbol{x}) = \begin{cases} \dfrac{[x_{(1)}]^2}{\prod_{i=1}^{n} {x_i}^2} , if\ x_{(1)} \leq \theta_0 \\ \dfrac{[\theta_0]^2}{\prod_{i=1}^{n} {x_i}^2} , if\ x_{(1)} > \theta_0 \end{cases}$$

$$\therefore\ \lambda(\boldsymbol{x}) = \frac{\sup_{\theta\in Ⓗ0} L(\theta|\boldsymbol{x})}{\sup_{\theta\in Ⓗ} L(\theta|\boldsymbol{x})} = \begin{cases} 1,\ if\ x_{(1)} \le \theta_0 \\ (\frac{\theta_0}{x_{(1)}})^n, if\ x_{(1)} > \theta_0 \end{cases}$$

We reject H_0 if $\lambda(\boldsymbol{x}) < c$, where c is a constant with $0 \le c < 1$

$$\Leftrightarrow (\frac{\theta_0}{x_{(1)}})^n < c\ if\ x_{(1)} > \theta_0 \Leftrightarrow x_{(1)} > \theta_0 c^{\frac{-1}{n}} > \theta_0$$

$\therefore \boldsymbol{R} = \{\boldsymbol{x}: x_{(1)} > \theta_0 c^{\frac{-1}{n}}\}$ is the rejection region.

$$\beta(\theta) = P_\theta(reject\ H_0)\ , \theta \in Ⓗ$$

$$= P_\theta\left(x_{(1)} > \theta_0 c^{\frac{-1}{n}}\right) = \left[P_\theta\left(x_1 > \theta_0 c^{\frac{-1}{n}}\right)\right]^n$$

$$= \left(\int_{\theta_0 c^{\frac{-1}{n}}}^{\infty} \theta x^{-2} I(x \ge \theta) dx\right)^n$$

$$= \begin{cases} \left(\int_{\theta_0 c^{\frac{-1}{n}}}^{\infty} \theta x^{-2} dx\right)^n, if\ \theta \le \theta_0 c^{\frac{-1}{n}} \\ \left(\int_{\theta}^{\infty} \theta x^{-2} dx\right)^n, if\ \theta > \theta_0 c^{\frac{-1}{n}} \end{cases}$$

$$= \begin{cases} (\frac{\theta}{\theta_0})^n c\ , if\ \theta \le c^{\frac{-1}{n}} \theta_0 \\ 1, if\ \theta > c^{\frac{-1}{n}} \theta_0 \end{cases}$$

$\Rightarrow \beta(\theta) = \begin{cases} (\frac{\theta}{\theta_0})^n c\ , if\ \theta \le c^{\frac{-1}{n}} \theta_0 \\ 1, if\ \theta > c^{\frac{-1}{n}} \theta_0 \end{cases}$ is the power function of θ.

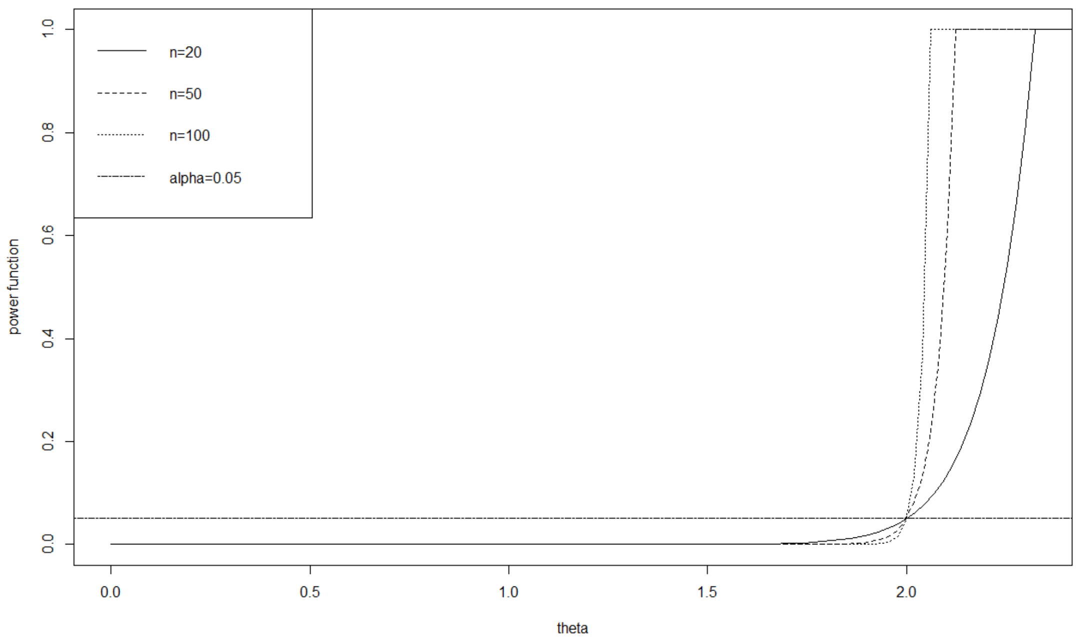
$$\therefore\ \alpha = \sup_{\theta\in Ⓗ0} \beta(\theta) = \sup_{0<\theta\le\theta_0} (\frac{\theta}{\theta_0})^n c = c$$

Hence $\boldsymbol{R} = \{\boldsymbol{x}: x_{(1)} > \theta_0 \alpha^{\frac{-1}{n}}\}$ is the LR-test with size α.

(c)

$\alpha = 0.05 = c$

We set $\theta_0 = 2$ $n = 20, 50, 100$ respectively



Apparently, the power is faster to reach the climax=1.0 as n is larger.

We could conclude that the power function is more powerful as n is larger.