#HW3.　　　103201506　數學 4A　邱奕豪

Problem1.

4.26

X and Y are independent random variables with X ~ exp(λ) and Y~ exp(μ).

It is impossible to obtain direct observations of X and Y. Instead ,we observe the random variables Z and W ,where

Z = min{X,Y}　　and　　$W=\begin{cases}1 & if\ \ Z=X\\ 0 & if\ \ Z=Y\end{cases}$

(This is a situation that arises, in particular , in medical experiments. The X and Y variables are censored.)

(a) Find the joint distribution of Z and W

(b) Prove that Z and W are independent.(Hint: show that P(Z≦z|W=i) = P(Z≦z) for i=0 or 1.)

Solution:

(a) P(Z≦z , W=1) = P(X≦z , X≦Y) = $\int_0^z\int_x^\infty \frac{1}{\lambda}\frac{1}{\mu}\ e^{\frac{-y}{\mu}}e^{\frac{-x}{\lambda}}$ dy dx

$= \int_0^z \frac{1}{\lambda}e^{-x(\frac{1}{\mu}+\frac{1}{\lambda})}$ dx = $\frac{\mu}{\lambda+\mu}(1-e^{-z(\frac{1}{\mu}+\frac{1}{\lambda})})$

Similarly , P(Z≦z , W=0) = P(Y≦z , Y≦X) = $\int_0^z\int_y^\infty \frac{1}{\lambda}\frac{1}{\mu}\ e^{\frac{-y}{\mu}}e^{\frac{-x}{\lambda}}$ dx dy

$= \int_0^z \frac{1}{\mu}e^{-y(\frac{1}{\mu}+\frac{1}{\lambda})}$ dy = $\frac{\lambda}{\lambda+\mu}\ (1-e^{-z(\frac{1}{\mu}+\frac{1}{\lambda})})$

So, f(z,w=1) = $\frac{d}{dz}$ P(Z≦z , W=1) = $\frac{1}{\mu}$ exp(-z($\frac{1}{\mu}+\frac{1}{\lambda}$))　　, z≧0

f(z,w=0) = $\frac{d}{dz}$ P(Z≦z , W=0) = $\frac{1}{\lambda}$ exp(-z($\frac{1}{\mu}+\frac{1}{\lambda}$))　　, z≧0

Hence , f(z,w) =$(\frac{1}{\lambda})^{1-w}\ (\frac{1}{\mu})^{w}$exp(-z($\frac{1}{\mu}+\frac{1}{\lambda}$))　　, z≧0 , w= 0 or 1

(b) P(Z≦z) = P(Z≦z , W=1) + P(Z≦z , W=0) = $1-e^{-z(\frac{1}{\mu}+\frac{1}{\lambda})}$

P(Z≦z | w= 1) = $\frac{P(Z\leqq z, W=1)}{P(w=1)} = \frac{P(Z\leqq z, W=1)}{p(X\leqq Y)}$

$= \frac{\frac{\mu}{\lambda+\mu}(1-e^{-z(\frac{1}{\mu}+\frac{1}{\lambda})})}{\frac{\mu}{\lambda+\mu}}$　　(P(X≦Y) =$\int_0^z\int_x^\infty \frac{1}{\lambda}\frac{1}{\mu}\ e^{\frac{-y}{\mu}}e^{\frac{-x}{\lambda}}$ dy dx = $\frac{\mu}{\lambda+\mu}$)

$= 1-e^{-z(\frac{1}{\mu}+\frac{1}{\lambda})}$　= P(Z≦z)

Similary , $P(Z \leqq z \mid w= 0) = P(Z \leqq z)$

So, Z and W are independent.

Problem2.

7.14

Let X and Y be independent exponential radom variables, with

$f(x \mid \lambda) = \frac{1}{\lambda} e^{\frac{-x}{\lambda}}$, x > 0 , $f(y \mid \mu) = \frac{1}{\mu} e^{\frac{-y}{\mu}}$, y > 0

we observe Z and W

Z = min{X,Y} and $W=\begin{cases} 1 & if \ \ Z = X \\ 0 & if \ Z = Y \end{cases}$

In exercise 4.26 the joint distribution of Z and W was obtained.Now assume that (Z_i,W_i) , i=1,…,n , are n iid observations. Find the MLEs of λ and μ

Solution:

$f(z,w) =(\frac{1}{\lambda})^{1-w} (\frac{1}{\mu})^{w}\exp(-z(\frac{1}{\mu}+\frac{1}{\lambda}))$, $z \geqq 0$, w= 0 or 1

let $T = ((Z_1,W_1),(Z_2,W_2),\ldots,(Z_n,W_n))$

$L(\mu, \lambda \mid T)= (\frac{1}{\lambda})^{n-\sum_{i=1}^{n} Wi} (\frac{1}{\mu})^{\sum_{i=1}^{n} Wi}\exp(-\sum_{i=1}^{n} Zi(\frac{1}{\mu}+\frac{1}{\lambda}))$

$\ln(L(\mu, \lambda \mid T)) = -(\sum_{i=1}^{n} Wi)\ln\mu - (n-\sum_{i=1}^{n} Wi)\ln \lambda - \sum_{i=1}^{n} Zi \, (\frac{1}{\mu}+\frac{1}{\lambda})$

$\frac{\partial}{\partial\mu} \ln(L(\mu, \lambda \mid T)) = \frac{-\sum_{i=1}^{n} Wi}{\mu} + \frac{\sum_{i=1}^{n} Zi}{\mu^2} \overset{set}{=} 0$

$\Rightarrow \hat{\mu} = \frac{\sum_{i=1}^{n} Zi}{\sum_{i=1}^{n} Wi}$

$\frac{\partial}{\partial\lambda} \ln(L(\mu, \lambda \mid T)) = \frac{-(n-\sum_{i=1}^{n} Wi)}{\lambda} + \frac{\sum_{i=1}^{n} Zi}{\lambda^2} \overset{set}{=} 0$

$\Rightarrow \hat{\lambda} = \frac{\sum_{i=1}^{n} Zi}{n-\sum_{i=1}^{n} Wi}$

$\frac{\partial^2}{\partial\lambda^2} \ln(L(\mu, \lambda \mid T)) = \frac{n-\sum_{i=1}^{n} Wi}{\lambda^2} - \frac{2\sum_{i=1}^{n} Zi}{\lambda^3}$

$\frac{\partial^2}{\partial\mu^2} \ln(L(\mu, \lambda \mid T)) = \frac{\sum_{i=1}^{n} Wi}{\mu^2} - \frac{2\sum_{i=1}^{n} Zi}{\mu^3}$

$\frac{\partial^2}{\partial\mu\,\partial\lambda} \ln(L(\mu, \lambda \mid T)) = 0$

$$H(\mu, \lambda) = \begin{pmatrix} \frac{\sum_{i=1}^{n} Wi}{\mu^2} - \frac{2\sum_{i=1}^{n} Zi}{\mu^3} & 0 \\ 0 & \frac{n-\sum_{i=1}^{n} Wi}{\lambda^2} - \frac{2\sum_{i=1}^{n} Zi}{\lambda^3} \end{pmatrix}$$

$$\Rightarrow \quad H(\hat{\mu}, \hat{\lambda}) = \begin{pmatrix} \frac{-\sum_{i=1}^{n} Wi}{\hat{\mu}^2} & 0 \\ 0 & \frac{-(n-\sum_{i=1}^{n} Wi)}{\hat{\lambda}^2} \end{pmatrix}$$

$$(x,y)\, H \binom{x}{y} = \frac{-\sum_{i=1}^{n} Wi}{\hat{\mu}^2} x^2 - \frac{n-\sum_{i=1}^{n} Wi}{\hat{\lambda}^2} y^2 \leqq 0$$

So, H is negative semidefinite

Hence, $\hat{\mu} = \frac{\sum_{i=1}^{n} Zi}{\sum_{i=1}^{n} Wi}$ $\hat{\lambda} = \frac{\sum_{i=1}^{n} Zi}{n-\sum_{i=1}^{n} Wi}$ is the mle estimator

Problem3.

Do the same exercise for the Weibull with the common shape parameter γ for X and T.

Solution :

$$f(x,\lambda,\gamma) = \frac{\gamma}{\lambda} (x/\lambda)^{\gamma-1} e^{-(\frac{x}{\lambda})^{\gamma}} \qquad , x \geqq 0$$

$$f(y\boldsymbol{\mu},\gamma) = \frac{\gamma}{\mu} (y/\lambda)^{\gamma-1} e^{-(\frac{y}{\mu})^{\gamma}} \qquad , y \geqq 0$$

$$P(Z \leqq z, W=0) = P(Z \leqq z, Y \leqq X) = \int_0^z \int_y^{\infty} \frac{\gamma}{\lambda} \frac{\gamma}{\mu} (x/\lambda)^{\gamma-1} (y/\lambda)^{\gamma-1} e^{-(\frac{y}{\mu})^{\gamma}} e^{-(\frac{x}{\lambda})^{\gamma}} dxdy =$$

$$\frac{\lambda^{\gamma}}{\mu^{\gamma}+\lambda^{\gamma}} (1-\exp(-Z^r(\frac{1}{\lambda^{\gamma}} + \frac{1}{\mu^{\gamma}}))$$

Similary, $P(Z \leqq z, W=1) = \frac{\mu^{\gamma}}{\mu^{\gamma}+\lambda^{\gamma}} (1-\exp(-Z^r(\frac{1}{\lambda^{\gamma}} + \frac{1}{\mu^{\gamma}}))$

$$f(z,w=0) = \frac{d}{dz} P(Z \leqq z, W=0) = \frac{rz^{r-1}}{\mu^{\gamma}} \exp(-Z^r(\frac{1}{\lambda^{\gamma}} + \frac{1}{\mu^{\gamma}})) \quad , z \geqq 0$$

$$f(z,w=1) = \frac{d}{dz} P(Z \leqq z, W=1) = \frac{\gamma z^{r-1}}{\lambda^{\gamma}} \exp(-Z^r(\frac{1}{\lambda^{\gamma}} + \frac{1}{\mu^{\gamma}})) \quad , z \geqq 0$$

$$f(z,w) = (\frac{rz^{r-1}}{\lambda^{\gamma}})^w (\frac{rz^{r-1}}{\mu^{\gamma}})^{1-w} \exp(-Z^r(\frac{1}{\lambda^{\gamma}} + \frac{1}{\mu^{\gamma}})) \qquad z \geqq 0, w= 0 \text{ or } 1$$

Then, check Z and W are independent.

Sol:

P(Z≦z) = P(Z≦z , W=1) + P(Z≦z , W=0) = 1-exp(-$Z^r(\frac{1}{\lambda^\gamma}+\frac{1}{\mu^\gamma})$)

P(Z≦z | w= 1) = $\frac{P(Z\leqq z, W=1)}{P(w=1)}$ = $\frac{P(Z\leqq z, W=1)}{p(X\leqq Y)}$

$$= \frac{\frac{\mu^\gamma}{\mu^\gamma+\lambda^\gamma}(1-\exp(-Z^r(\frac{1}{\lambda^\gamma}+\frac{1}{\mu^\gamma})))}{\frac{\mu^\gamma}{\mu^\gamma+\lambda^\gamma}}$$

(because P(X≦Y) =$\int_0^\infty\int_x^\infty \frac{\gamma}{\lambda}\frac{\gamma}{\mu}$ $(x/\lambda)^{\gamma-1}(y/\lambda)^{\gamma-1}$ $e^{-(\frac{y}{\mu})^\gamma}e^{-(\frac{x}{\lambda})^\gamma}$ dydx = $\frac{\mu^\gamma}{\mu^\gamma+\lambda^\gamma}$)

= 1-exp(-$Z^r(\frac{1}{\lambda^\gamma}+\frac{1}{\mu^\gamma})$)

= P(Z≦z)

Similary , P(Z≦z | w= 0) = P(Z≦z)

So, Z and W are independent.

Then, find the MLEs of λ and μ and γ.

Sol :

let T = ((Z_1,W_1),(Z_2,W_2),….,(Z_n,W_n))

$$L(\lambda,\boldsymbol{\mu},\gamma|(Z_n,W_n)) = \prod_{i=1}^n((\frac{rzi^{r-1}}{\lambda^\gamma})^{wi}) \prod_{i=1}^n((\frac{rzi^{r-1}}{\mu^\gamma})^{1-wi}) \exp(-\sum_{i=1}^n Zi^\gamma(\frac{1}{\lambda^\gamma}+\frac{1}{\mu^\gamma}))$$

ln(L(λ,$\boldsymbol{\mu}$,γ|(Z_n,W_n)))

$$=\sum_{i=1}^n(1-Wi)\ln(\frac{rzi^{r-1}}{\mu^\gamma}) +\sum_{i=1}^n Wi \ln(\frac{rzi^{r-1}}{\lambda^\gamma}) -\sum_{i=1}^n Zi^\gamma(\frac{1}{\lambda^\gamma}+\frac{1}{\mu^\gamma})$$

$$\frac{\partial}{\partial\lambda}\ln(L(\lambda,\boldsymbol{\mu},\gamma|(Z_n,W_n))) = \frac{-\gamma}{\lambda}\sum_{i=1}^n Wi + \frac{\gamma}{\lambda^{\gamma+1}}\sum_{i=1}^n Zi^\gamma =^{set} 0$$

$$\Rightarrow \hat{\lambda} = \sqrt[\hat{\gamma}]{\frac{\sum_{i=1}^n Zi^{\hat{\gamma}}}{\sum_{i=1}^n Wi}}$$

$$\frac{\partial}{\partial\mu}\ln(L(\lambda,\boldsymbol{\mu},\gamma|(Z_n,W_n))) = \frac{-\gamma}{\mu}\sum_{i=1}^n(1-Wi) + \frac{\gamma}{\mu^{\gamma+1}}\sum_{i=1}^n Zi^\gamma =^{set} 0$$

$$\Rightarrow \hat{\boldsymbol{\mu}} = \sqrt[\hat{\gamma}]{\frac{\sum_{i=1}^n Zi^{\hat{\gamma}}}{\sum_{i=1}^n(1-Wi)}}$$

$$\frac{\partial}{\partial\gamma}\ln(L(\lambda,\boldsymbol{\mu},\gamma|(Z_n,W_n))) = \ln\lambda\sum_{i=1}^n Wi+\frac{n}{\gamma}+\sum_{i=1}^n lnZi - n\ln\boldsymbol{\mu} - \ln\boldsymbol{\mu}\sum_{i=1}^n Wi =^{set} 0$$

⇨ $\hat{\gamma} = \dfrac{n}{n\ln\hat{\mu} - \sum_{i=1}^{n} \ln Zi - \ln\frac{\hat{\mu}}{\hat{\lambda}} \sum_{i=1}^{n} Wi}$

Problem4.

exercise 7.50 (a) Details including the calculation of E(S)= , (b) Details including the calculation of a , (c) Detailed formulas to apply the Factorization thm and verify the complereness

Solution :

(a)

Note: $\frac{(n-1)S^2}{\theta^2}$ ~ χ^2(n-1)

Let T = $\frac{(n-1)S^2}{\theta^2}$

$E(\sqrt{T}) = \int_0^\infty t^{\frac{1}{2}} \dfrac{(\frac{1}{2})^{\frac{n-1}{2}} t^{\frac{n-1}{2}-1} e^{\frac{-t}{2}}}{\Gamma(\frac{n-1}{2})} dt$

$= \dfrac{(\frac{1}{2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^\infty t^{\frac{n}{2}-1} e^{\frac{-t}{2}} dt$

$= \dfrac{(\frac{1}{2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \times \dfrac{\Gamma(\frac{n}{2})}{(\frac{1}{2})^{\frac{n}{2}}}$

$= \dfrac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\sqrt{2}}$

⇨ $E\left(\frac{\sqrt{n-1}\,S}{\theta}\right) = \dfrac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\sqrt{2}}$

⇨ $E(s) = \dfrac{\theta}{\sqrt{n-1}} \dfrac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\sqrt{2}}$

Then, $E(a\bar{x} + (1-a)CS) = aE(\bar{x}) + (1-a)CE(S)$

$= a\,\theta + (1-a)\theta$

$= \theta$

(b)

$Var(a\bar{x} + (1-a)CS) = a^2 var(\bar{x}) + (1-a)^2C^2 var(S) + cov(a\bar{x}, (1-a)CS)$

$= a^2 \frac{\theta^2}{n} + (1-a)^2C^2 \left(E(S^2) - (E(S))^2\right)$

$= \frac{a\theta^2}{n} + (1-a)^2C^2 \left(\theta^2 - \frac{\theta^2}{c^2}\right)$

$= \left(\frac{a^2}{n} + (1-a)^2c^2 - (1-a)^2\right)\theta^2$

⇨ $\min\{\frac{a^2}{n}+(1-\mathrm{a})^2c^2-\ (1-\mathrm{a})^2\ \}$

⇨ $\frac{\mathrm{d}}{\mathrm{da}}(\frac{a^2}{n}+(1-\mathrm{a})^2c^2-\ (1-\mathrm{a})^2\ ) =^{set}\ 0$

⇨ $\hat{\mathrm{a}} = \frac{nc^2-n}{1+nc^2-n}$

$\frac{\mathrm{d}^2}{\mathrm{d}a^2}(\frac{a^2}{n}+(1-\mathrm{a})^2c^2-\ (1-\mathrm{a})^2) = \frac{2}{n} + 2c^2-2 \quad > \quad 0$ (because $c^2 > 1\ for\ all\ n \geqq 2$)

⇨ So, $\hat{\mathrm{a}} = \frac{nc^2-n}{1+nc^2-n}$ produces the estimator with minimum variance

(c)

$f(x|\theta) = \frac{1}{\theta^2\sqrt{2\pi}}\ \exp(\frac{-1}{2\theta^2}(xi-\theta)^2\ )$

$L(\theta|X) = (\frac{1}{2\pi\theta^2})^{\frac{n}{2}}\ \exp(\frac{-1}{2\theta^2}\sum_{i=1}^{n}(xi-\theta)^2)$

$= (\frac{1}{2\pi})^{\frac{n}{2}}\ (\frac{1}{\theta})^n\ \exp(\frac{-1}{2\theta^2}(\sum_{i=1}^{n}(xi-\bar{x})^2 - n(\theta-\bar{x})^2))$

$= (\frac{1}{2\pi})^{\frac{n}{2}}\ (\frac{1}{\theta})^n\ \exp(\frac{-(n-1)S^2}{2\theta^2} + \frac{n(\theta-\bar{x})^2}{2\theta^2})$

Let $h(x) = (\frac{1}{2\pi})^{\frac{n}{2}}$ and $g(T(x), \theta) = (\frac{1}{\theta})^n\ \exp(\frac{-(n-1)S^2}{2\theta^2} + \frac{n(\theta-\bar{x})^2}{2\theta^2})$

By factorization thm, ($\bar{x}$,S^2) is sufficient statistics

$E(\bar{X}) = \theta$ and $E(S)\ \frac{\theta}{c}$

Let g(x) = $\bar{X}$ - CS

⇨ $E(g(x)) = E(\bar{x}) - E(CS) = \theta - \theta = 0$

But g(x) = $\bar{X}$ - CS $\neq$ 0

So, ($\bar{x}$,S^2) is not a complete sufficient statistics.

Problem5.

exercise 7.51(a)-(d). Prove your answer by formulas (not just words).

Solution :

(a)

$E(\theta - a1\bar{x} - a2(CS))^2$ = var$(\theta - a1\bar{x} - a2(CS))$ + $(E(\theta - a1\bar{x} - a2(CS)))^2$

= $(a1)^2$ var($\bar{x}$) + $(a2)^2c^2$var(S) + θ^2 $(a1 + a2 - 1)^2$

= $\frac{a1^2}{n}\theta^2 + (a2c)^2 - a2^2\theta^2 + \theta^2(a1 + a2 - 1)^2$

= $(\frac{a1^2}{n} + (a2c)^2 - a2^2\theta^2 + (a1 + a2 - 1)^2)\ \theta^2$

⇨ min$(\frac{a1^2}{n} + (a2c)^2 - a2^2\theta^2 + (a1 + a2 - 1)^2)$

⇨ $\frac{\partial}{\partial a1}(\frac{a1^2}{n} + (a2c)^2 - a2^2\theta^2 + (a1 + a2 - 1)^2) = \frac{2a1}{n} + 2(a_1+a_2-1) =^{set} 0$

⇨ $\frac{\partial}{\partial a2}(\frac{a1^2}{n} + (a2c)^2 - a2^2\theta^2 + (a1 + a2 - 1)^2) = 2a_2c^2 + 2(a_1+a_2-1) - 2a_2 =^{set} 0$

⇨ $\begin{cases} a1 + a1n + a2n - n = 0 \\ a2c^2 + a1 - 1 = 0 \end{cases}$

⇨ $\hat{a_1} = 1 - \frac{c^2}{(n+1)c^2 - n}$ $\hat{a_2} = \frac{1}{(n+1)c^2 - n}$

(b)

$B^2(T^*) = (E(T^* - \theta))^2 = (a1 + a2 - 1)^2\ \theta^2$

$Var(T^*) = (\frac{a1^2}{n} + a2^2\ (c^2 - 1))\ \theta^2$

$MSE(T^*) = ((a1 + a2 - 1)^2 + (\frac{a1^2}{n} + a2^2\ (c^2 - 1)))\ \theta^2$

$= (\frac{(c^2-1)^2 + (c^2-1) + n(c^2-1)^2}{((n+1)c^2-n)^2})\ \theta^2$

$= (\frac{(c^2-1)(c^2(n+1)-n)}{((n+1)c^2-n)^2})\ \theta^2$

$E(T) = \theta$ $B^2(T) = 0$

$Var(T) = (\frac{a^2}{n} + (1 - a)^2c^2 - (1 - a)^2)\ \theta^2$

$\text{MSE(T)} = (\frac{a^2}{n}+(1-\text{a})^2\text{c}^2-(1-a)^2)\ \theta^2$

$= \{ (\frac{n(c^2-1)^2}{(1+nc^2-n)^2}) +(1-\text{a})^2(c^2-1) \}\ \theta^2$

$= (\frac{n(c^2-1)^2+c^2-1}{(1+nc^2-n)^2})\ \theta^2$

$= (\frac{(c^2-1)(n(c^2-1)+1)}{(1+nc^2-n)^2})\theta^2$

$\frac{MSE(T*)}{MSE(T)} = \frac{(\frac{(c^2-1)(c^2(n+1)-n)}{((n+1)c^2-n)^2})\ \theta^2}{(\frac{(c^2-1)(n(c^2-1)+1)}{(1+nc^2-n)^2})\theta^2} = \frac{(c^2(n+1)-n)(1+n(c^2-1))^2}{((n+1)c^2-n)^2(n(c^2-1)+1)}$

$= \frac{n(c^2-1)+1}{(n+1)c^2-n}$

$= \frac{n(c^2-1)+1}{n(c^2-1)+c^2} < 1$ (because $c^2 > 1\ for\ all\ n > 2$)

So, the MSE of T* is smaller than the MSE of the T .

(c)

$F_{T^{*+}}(t) = P(T^{*+} \leqq t)$

$= P(\max\{0,T^*\}\leqq t)$

$= \begin{cases} 0 & t < 0 \\ P(T* \leqq 0) & t = 0 \\ P(T* \leqq t) & t > 0 \end{cases}$

$E(\theta-T^{*+})^2 = (\theta-0)^2\ P(T^{*+}=0) + \int_0^\infty (\theta-t)^2\ f_{T^*}(t)dt$

$= \theta^2\ P(T^*\leqq 0) + \int_0^\infty (\theta-t)^2\ f_{T^*}(t)dt$

$= \int_{-\infty}^0 \theta^2\ f_{T^*}(t)dt + \int_0^\infty (\theta-t)^2\ f_{T^*}(t)dt$

$\leqq \int_{-\infty}^0 (\theta-t)^2\ f_{T^*}(t)dt + \int_0^\infty (\theta-t)^2\ f_{T^*}(t)dt$

$= \int_{-\infty}^\infty (\theta-t)^2\ f_{T^*}(t)dt$

$= E(\theta-T^*)^2$

So, MSE of T^{*+} is smaller than the mse of T*

(d)

$f(X|\theta) = \frac{1}{\theta\sqrt{2\pi}}\ \exp(\frac{-1}{2\theta^2}(\text{x}-\theta)^2)$

because this pdf does not fit the definition of a location parameter or scale

parameter.
So, θ can’t be classified as a location parameter or scale parameter.