HW#3, Due on 10/13(Fri), Submit to TA during 10:00-10:30

- **Exercise 4.26**

Solution (a).

Let X and Y be independent exponential random variables with probability density functions defined as

$$f_\lambda(x)=\frac{1}{\lambda}e^{-x/\lambda},\quad \lambda>0,\ x>0,\qquad f_\mu(y)=\frac{1}{\mu}e^{-y/\mu},\quad \mu>0,\ y>0.$$

Under certain situations, one cannot obtain direct observations of X and Y. Instead, one observes random variables Z and W, where

$$Z=\min(X,Y),\qquad W=\begin{cases}1 & \text{if } Z=X,\\ 0 & \text{if } Z=Y.\end{cases}$$

This setting is known as "competing risks". Now, we derive the joint distribution function of Z and W. By straightforward calculations,

$$\begin{aligned}\Pr(Z\le z,W=0)&=\Pr(\min(X,Y)\le z,\min(X,Y)=Y)=\Pr(Y\le z,Y\le X)\\&=\int_0^z\frac{1}{\mu}e^{-y/\mu}\int_y^\infty\frac{1}{\lambda}e^{-x/\lambda}dxdy=\int_0^z\frac{1}{\mu}\exp\left\{-\frac{(\lambda+\mu)y}{\lambda\mu}\right\}dy\\&=\frac{\lambda}{\lambda+\mu}\left[1-\exp\left\{-\frac{(\lambda+\mu)z}{\lambda\mu}\right\}\right].\end{aligned}$$

In a similar fashion,

$$\begin{aligned}\Pr(Z\le z,W=1)&=\Pr(\min(X,Y)\le z,\min(X,Y)=X)=\Pr(X\le z,X\le Y)\\&=\frac{\mu}{\lambda+\mu}\left[1-\exp\left\{-\frac{(\lambda+\mu)z}{\lambda\mu}\right\}\right].\end{aligned}$$

Solution (b).

We first obtain the marginal distributions of Z and W. By straightforward calculations,

$$\Pr(W=0)=\Pr(Z\le\infty,W=0)=\frac{\lambda}{\lambda+\mu},\qquad \Pr(W=1)=\Pr(Z\le\infty,W=1)=\frac{\mu}{\lambda+\mu}.$$

$$\Pr(Z\le z)=\Pr(Z\le z,W=0)+\Pr(Z\le z,W=1)=1-\exp\left\{-\frac{(\lambda+\mu)z}{\lambda\mu}\right\}.$$

Therefore, one obtains the identities

$$\Pr(Z\le z,W=j)=\Pr(Z\le z)\Pr(W=j),\qquad z>0,\ j=0,1.$$

Hence we have shown the independence of Z and W.

- **Exercise 7.14**

Following Exercise 4.26, now we assume that (Z_i, W_i), $i = 1, 2, \cdots, n$ are independent and identical distributed random variables. We aim to find the MLEs of λ and μ.

As we mentioned in Exercise 4.26, this question is based on competing risks. To make things clear, we briefly introduce some important concepts of competing risks model. In Exercise 4.26 (a), the two probabilities $\Pr(Z \le z, W = j)$, $j = 0, 1$ are the so-called "sub-distribution functions" and we define $F_{\lambda,\mu}(z, j) = \Pr(Z \le z, W = j)$, $j = 0, 1$. In addition, the so-called "sub-density functions" are defined as

$$f_{\lambda,\mu}(z, j) = \frac{\partial}{\partial z} F_{\lambda,\mu}(z, j), \qquad j = 0, 1.$$

To be specific, we have

$$f_{\lambda,\mu}(z, j) = \frac{\partial}{\partial z} F_{\lambda,\mu}(z, j) = \frac{1}{\lambda^j \mu^{1-j}} \exp\left\{ -\frac{(\lambda + \mu) z}{\lambda \mu} \right\}, \qquad j = 0, 1.$$

The likelihood function (Lawless 2003, p.435) is

$$L_n(\lambda, \mu) = \prod_{i=1}^{n} f_{\lambda,\mu}(Z_i, 1)^{W_i} f_{\lambda,\mu}(Z_i, 0)^{1-W_i} .$$

Thus, the log-likelihood function is

$$\begin{aligned} \ell_n(\lambda, \mu) &= \log L_n(\lambda, \mu) = \sum_{i=1}^{n} W_i \log f_{\lambda,\mu}(Z_i, 1) + \sum_{i=1}^{n} (1 - W_i) \log f_{\lambda,\mu}(Z_i, 0) \\ &= -\log \mu \left(n - \sum_{i=1}^{n} W_i \right) - \log \lambda \sum_{i=1}^{n} W_i - \frac{1}{\lambda} \sum_{i=1}^{n} Z_i - \frac{1}{\mu} \sum_{i=1}^{n} Z_i. \end{aligned}$$

The MLEs can be obtained by solving $\partial \ell_n(\lambda, \mu) / \partial \lambda = 0$ and $\partial \ell_n(\lambda, \mu) / \partial \mu = 0$ which are equivalent to

$$-\frac{1}{\lambda} \sum_{i=1}^{n} W_i + \frac{1}{\lambda^2} \sum_{i=1}^{n} Z_i = 0, \qquad -\frac{1}{\mu} \left(n - \sum_{i=1}^{n} W_i \right) + \frac{1}{\mu^2} \sum_{i=1}^{n} Z_i = 0.$$

Clearly, the MLEs are

$$\hat{\lambda} = \sum_{i=1}^{n} Z_i \left(\sum_{i=1}^{n} W_i \right)^{-1}, \qquad \hat{\mu} = \sum_{i=1}^{n} Z_i \left(n - \sum_{i=1}^{n} W_i \right)^{-1}, \qquad \sum_{i=1}^{n} W_i \ne 0 \text{ or } n.$$

The MLEs attain the maximum of log-likelihood function is ensured by examining the Hessian matrix. Since the off-diagonal elements of the Hessian matrix are 0, it suffices to show

$$\left. \frac{\partial^2}{\partial \lambda^2} \ell_n(\lambda, \mu) \right|_{\lambda = \hat{\lambda}} = -\left(\sum_{i=1}^{n} W_i \right)^3 \left(\sum_{i=1}^{n} Z_i \right)^{-2} < 0, \quad \left. \frac{\partial^2}{\partial \mu^2} \ell_n(\lambda, \mu) \right|_{\mu = \hat{\mu}} = -\left(n - \sum_{i=1}^{n} W_i \right)^3 \left(\sum_{i=1}^{n} Z_i \right)^{-2} < 0.$$

- **Do the same exercise for the Weibull with the common shape parameter γ for *X* and *Y*.**

Solution.

Let X and Y be independent Weibull random variables which share the common shape parameter $\gamma > 0$. Their probability density functions are defined as

$$f_{\lambda,\gamma}(x) = \frac{\gamma}{\lambda} x^{\gamma-1} e^{-x^{\gamma}/\lambda}, \quad \lambda > 0, \ x > 0, \qquad f_{\mu,\gamma}(y) = \frac{\gamma}{\mu} y^{\gamma-1} e^{-y^{\gamma}/\mu}, \quad \mu > 0, \ y > 0.$$

Similarly, we define the random variables Z and W as in Exercise 7.14. By straightforward calculations, one has

$$F_{\lambda,\mu,\gamma}(z,0) = \Pr(Y \le z, Y \le X) = \int_0^z \frac{\gamma}{\mu} y^{\gamma-1} e^{-y^{\gamma}/\mu} \int_y^{\infty} \frac{\gamma}{\lambda} x^{\gamma-1} e^{-x^{\gamma}/\lambda} dx dy$$

$$= \int_0^z \frac{\gamma}{\mu} y^{\gamma-1} \exp\left\{ -\frac{(\lambda+\mu) y^{\gamma}}{\lambda\mu} \right\} dy = \frac{\lambda}{\lambda+\mu} \left[1 - \exp\left\{ -\frac{(\lambda+\mu) z^{\gamma}}{\lambda\mu} \right\} \right].$$

$$F_{\lambda,\mu,\gamma}(z,1) = \Pr(Z \le z, W = 1) = \Pr(X \le z, X \le Y) = \frac{\mu}{\lambda+\mu} \left[1 - \exp\left\{ -\frac{(\lambda+\mu) z^{\gamma}}{\lambda\mu} \right\} \right].$$

The marginal distributions of Z and W are

$$\Pr(W = 0) = \Pr(Z \le \infty, W = 0) = \frac{\lambda}{\lambda+\mu}, \qquad \Pr(W = 1) = \Pr(Z \le \infty, W = 1) = \frac{\mu}{\lambda+\mu}.$$

$$\Pr(Z \le z) = \Pr(Z \le z, W = 0) + \Pr(Z \le z, W = 1) = 1 - \exp\left\{ -\frac{(\lambda+\mu) z}{\lambda\mu} \right\}.$$

Therefore, one obtains the identities

$$\Pr(Z \le z, W = j) = \Pr(Z \le z) \Pr(W = j), \qquad z > 0, \ j = 0, 1.$$

Hence we have shown the independence of Z and W.

The joint probability density is

$$f_{\lambda,\mu,\gamma}(z, j) = \frac{\partial}{\partial z} F_{\lambda,\mu,\gamma}(z, j) = \frac{\gamma z^{\gamma-1}}{\lambda^{j} \mu^{1-j}} \exp\left\{ -\frac{(\lambda+\mu) z^{\gamma}}{\lambda\mu} \right\}, \qquad j = 0, 1.$$

Thus, the log-likelihood function is

$$\ell_n(\lambda, \mu, \gamma) = \sum_{i=1}^{n} W_i \log f_{\lambda,\mu,\gamma}(Z_i, 1) + \sum_{i=1}^{n} (1 - W_i) \log f_{\lambda,\mu,\gamma}(Z_i, 0)$$

$$= n \log \gamma + (\gamma - 1) \sum_{i=1}^{n} \log Z_i - \log \lambda \sum_{i=1}^{n} W_i - \log \mu \left(n - \sum_{i=1}^{n} W_i \right) - \frac{1}{\lambda} \sum_{i=1}^{n} Z_i^{\gamma} - \frac{1}{\mu} \sum_{i=1}^{n} Z_i^{\gamma}.$$

The MLEs can be obtained by solving $\partial \ell_n(\lambda,\mu,\gamma)/\partial\lambda = 0$, $\partial \ell_n(\lambda,\mu,\gamma)/\partial\mu = 0$ and $\partial \ell_n(\lambda,\mu,\gamma)/\partial\gamma = 0$ which are equivalent to

$$-\frac{1}{\lambda}\sum_{i=1}^{n} W_i + \frac{1}{\lambda^2}\sum_{i=1}^{n} Z_i^{\gamma} = 0, \qquad -\frac{1}{\mu}\left(n - \sum_{i=1}^{n} W_i\right) + \frac{1}{\mu^2}\sum_{i=1}^{n} Z_i^{\gamma} = 0,$$

$$\frac{n}{\gamma} + \sum_{i=1}^{n}\log Z_i - \frac{1}{\lambda}\sum_{i=1}^{n} Z_i^{\gamma}\log Z_i - \frac{1}{\mu}\sum_{i=1}^{n} Z_i^{\gamma}\log Z_i = 0.$$

With some further simplifications, we obtain

$$\hat{\lambda} = \sum_{i=1}^{n} Z_i^{\hat{\gamma}}\left(\sum_{i=1}^{n} W_i\right)^{-1}, \qquad \hat{\mu} = \sum_{i=1}^{n} Z_i^{\hat{\gamma}}\left(n - \sum_{i=1}^{n} W_i\right)^{-1},$$

$$\hat{\gamma} = \left(\sum_{i=1}^{n} Z_i^{\hat{\gamma}}\log Z_i \Big/ \sum_{i=1}^{n} Z_i^{\hat{\gamma}} - \frac{1}{n}\sum_{i=1}^{n}\log Z_i\right)^{-1}.$$

The expression for $\hat{\gamma}$ is obtained by plugging

$$\lambda = \sum_{i=1}^{n} Z_i^{\gamma}\left(\sum_{i=1}^{n} W_i\right)^{-1}, \qquad \mu = \sum_{i=1}^{n} Z_i^{\gamma}\left(n - \sum_{i=1}^{n} W_i\right)^{-1},$$

into $\partial \ell_n(\lambda,\mu,\gamma)/\partial\gamma = 0$.

Since the closed-form of the MLEs are not available, one has to perform some numerical methods to obtain the MLEs. Here, we suggest applying the fixed-point iteration method to obtain the MLE $\hat{\gamma}$, then the MLEs $\hat{\lambda}$ and $\hat{\mu}$ are obtained by using the above formulas. Now, we state the fixed-point iteration algorithm.

Algorithm 1 Fixed-point iteration algorithm

Step 1. Set initial value $\gamma^{(0)}$.

Step 2. Repeat the fixed-point iteration:

$$\gamma^{(k+1)} = \left(\sum_{i=1}^{n} Z_i^{\gamma^{(k)}}\log Z_i \Big/ \sum_{i=1}^{n} Z_i^{\gamma^{(k)}} - \frac{1}{n}\sum_{i=1}^{n}\log Z_i\right)^{-1}.$$

- If $|\gamma^{(k+1)} - \gamma^{(k)}| < 10^{-5}$, stop the algorithm and set the MLE as $\gamma^{(k+1)}$.

For illustration, we generate random samples (X_i, Y_i), $i = 1, 2, \cdots, 300$ from the Weibull distributions with true parameter $(\lambda, \mu, \gamma) = (1, 2, 3)$. Then, we obtain the competing risks data (Z_i, W_i), $i = 1, 2, \cdots, 300$, where $Z_i = \min(X_i, Y_i)$ and $W_i = \mathbf{I}(Z_i = X_i)$. Based on the generated data, we apply Algorithm 1 with initial value $\gamma^{(0)} = 1$ and it converges in 27 iterations. The result of estimation is $\hat{\gamma} = 2.8751$, then we obtain $\hat{\lambda} = 0.9714$ and $\hat{\mu} = 2.0024$. Figure 1 reveals that the MLEs attain the maximum of the log-likelihood function. R codes are available in Appendix 1.

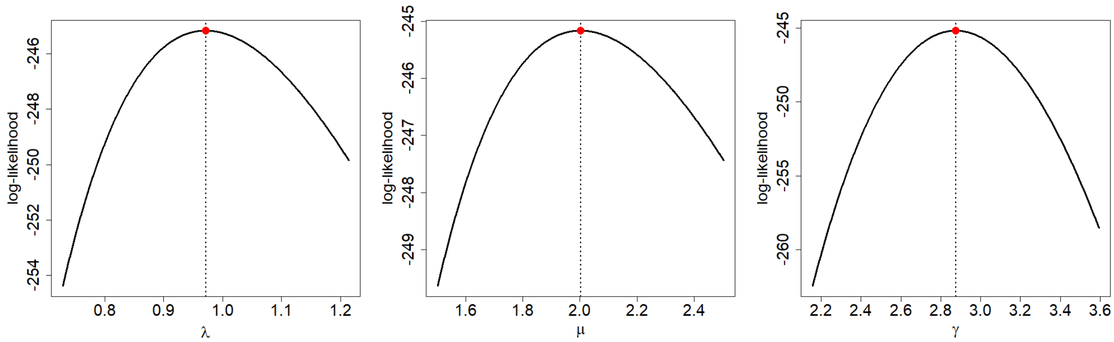

Figure 1. Log-likelihood functions under the Weibull competing risks model based on the generated data. The vertical lines are drawn at $\hat{\lambda} = 0.9714$, $\hat{\mu} = 2.0024$, $\hat{\gamma} = 2.8751$

- **Exercise 7.50 (a) Details including the calculation of E[S]=, (b) Details including the calculation of *a*, (c) Detailed <u>formulas</u> to apply the Factorization theorem and verify the completeness.**

Solution (a).

Let $X_1, \cdots, X_n$ be independent and identical distributed random variables from $N(\theta, \theta^2)$, $\theta > 0$. We define

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{and} \qquad S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 .$$

Clearly, $\overline{X}_n$ is an unbiased estimator for θ. We aim to find another unbiased estimator for θ. One may define $Y = (n-1)S_n^2/\theta^2$ hence $Y \sim \chi^2_{\mathrm{df}=n-1} = \Gamma((n-1)/2, 2)$. Then one obtains

$$\begin{aligned} E(S_n) &= E\{(S_n^2)^{1/2}\} = \frac{1}{\sqrt{n-1}} E\{(\theta^2 Y)^{1/2}\} \\ &= \frac{1}{\sqrt{n-1}}\int_0^\infty \theta y^{1/2} \frac{1}{\Gamma\{(n-1)/2\}2^{(n-1)/2}} y^{(n-1)/2-1} e^{-y/2} dy \\ &= \frac{\sqrt{2}\Gamma(n/2)}{\sqrt{n-1}\Gamma\{(n-1)/2\}}\theta. \end{aligned}$$

This implies

$$E(cS_n) = \theta, \qquad c = \frac{\sqrt{n-1}\Gamma\{(n-1)/2\}}{\sqrt{2}\Gamma(n/2)} .$$

We find that cS_n is also an unbiased estimator for θ. Then,

$$E\{a\overline{X}_n + (1-a)cS_n\} = aE(\overline{X}_n) + E(cS_n) - aE(cS_n) = a\theta + \theta - a\theta = \theta .$$

Hence we have shown the desired result.

Solution (b).

We have

$$\mathrm{var}\{a\overline{X}_n + (1-a)cS_n\} = a^2\,\mathrm{var}(\overline{X}_n) + (1-a)^2\,\mathrm{var}(cS_n),$$

where $\mathrm{var}(\overline{X}_n) = \theta^2/n$ and $\mathrm{var}(cS_n) = E(c^2 S_n^2) - \{E(cS_n)\}^2 = (c^2-1)\theta^2$. To minimize the variance, one has to solve $\partial\,\mathrm{var}\{a\overline{X}_n + (1-a)cS_n\}/\partial a = 0$ which is equivalent to

$$2a\,\mathrm{var}(\overline{X}_n) - 2(1-a)\,\mathrm{var}(S_n) = 0 .$$

Obviously, the solution is

$$a = \frac{\mathrm{var}(S_n)}{\mathrm{var}(\overline{X}_n) + \mathrm{var}(S_n)} = \frac{(c^2-1)\theta^2}{\theta^2/n + (c^2-1)\theta^2} = \frac{n(c^2-1)}{n(c^2-1)+1} .$$

Solution (c).

The probability density function of $N(\theta,\theta^2)$ is defined as

$$f_\theta(x)=\frac{1}{\sqrt{2\pi\theta^2}}\exp\left\{-\frac{(x-\theta)^2}{2\theta^2}\right\},\qquad -\infty<x<\infty,\ \ \theta>0.$$

Then, the joint probability density function is

$$\begin{aligned} f_\theta(x_1,\cdots,x_n)&=\prod_{i=1}^{n} f_\theta(x_i)=\left(\frac{1}{2\pi\theta^2}\right)^{n/2}\exp\left\{-\frac{1}{2\theta^2}\sum_{i=1}^{n}(x_i-\theta)^2\right\}\\ &=\left(\frac{1}{2\pi}\right)^{n/2}\frac{1}{\theta^n}\exp\left\{-\frac{1}{2\theta^2}\sum_{i=1}^{n}x_i^2+\frac{1}{\theta}\sum_{i=1}^{n}x_i-\frac{n}{2}\right\}\\ &=g_\theta\{T_1(x_1,\cdots,x_n),T_2(x_1,\cdots,x_n)\}h(x_1,\cdots,x_n), \end{aligned}$$

where $T_1(x_1,\cdots,x_n)=\Sigma_{i=1}^n x_i^2$, $T_2(x_1,\cdots,x_n)=\Sigma_{i=1}^n x_i$,

$$g_\theta(t_1,t_2)=\frac{1}{\theta^n}\exp\left(-\frac{t_1}{2\theta^2}+\frac{t_2}{\theta}\right),\qquad h(x_1,\cdots,x_n)=\left(\frac{1}{2\pi}\right)^{n/2}e^{-n/2}.$$

By the factorization theorem, $(T_1,T_2)=(\Sigma_{i=1}^n X_i,\Sigma_{i=1}^n X_i^2)$ is a sufficient statistics. Then,

$$(\bar{X}_n,S_n^2)=\left(\frac{1}{n}\sum_{i=1}^{n}X_i,\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\right)$$

is a function of (T_1,T_2) hence it is also a sufficient statistics.

To show the incompleteness, we have $E(\bar{X}_n-cS_n)=\theta-\theta=0$ for all $\theta>0$. Thus, we have found a non-zero function $\bar{X}_n-cS_n$ with its expectation always equals to zero. Therefore, the sufficient statistics $(\bar{X}_n,S_n^2)$ is not complete.

- **Exercise 7.51 (a)-(d).** **<u>Prove your answer by formulas</u> (not just words).**

  Solution (a).

  Consider a class of estimator

  $$\mathcal{T} = \{ T : T = a_1 \overline{X}_n + a_2 (cS_n) \},$$

  where we do not assume $a_1 + a_2 = 1$. We aim to find the estimator $T \in \mathcal{T}$ that minimize the mean square error $E_\theta\{ (T - \theta)^2 \}$, call it T^*. By straightforward calculations, since $a_1 \overline{X}_n + a_2 (cS_n)$ may be biased, we obtain

  $$\begin{aligned} E_\theta[\{ a_1 \overline{X}_n + a_2 (cS_n) - \theta \}^2] &= a_1^2 \operatorname{var}(\overline{X}_n) + a_2^2 c^2 \operatorname{var}(S_n) + (a_1 + a_2 - 1)^2 \theta^2 \\ &= \frac{a_1^2}{n} \theta^2 + a_2^2 (c^2 - 1) \theta^2 + (a_1 + a_2 - 1)^2 \theta^2. \end{aligned}$$

  To minimize the mean square error, one has to solve $\partial E_\theta[\{ a_1 \overline{X}_n + a_2 (cS_n) - \theta \}^2] / \partial a_1 = 0$ and $\partial E_\theta[\{ a_1 \overline{X}_n + a_2 (cS_n) - \theta \}^2] / \partial a_2 = 0$ which are equivalent to

  $$2\left(\frac{n+1}{n} \right) a_1 + 2(a_2 - 1) = 0, \qquad 2a_2 c^2 + 2(a_1 - 1) = 0.$$

  The solutions are

  $$a_1 = \frac{n(c^2 - 1)}{n(c^2 - 1) + c^2}, \qquad a_2 = \frac{1}{n(c^2 - 1) + c^2}.$$

  Note that the Hessian matrix is positive definite

  $$2\begin{bmatrix} (n+1) / n & 1 \\ 1 & c^2 \end{bmatrix} > 0.$$

  Thus, we obtain the estimator

  $$T^* = \frac{n(c^2 - 1)}{n(c^2 - 1) + c^2} \overline{X}_n + \frac{1}{n(c^2 - 1) + c^2} (cS_n)$$

  minimize the mean square error.

  Solution (b).

  The mean square error of T^* is

  $$\begin{aligned} E_\theta\{ (T^* - \theta)^2 \} &= \frac{n(c^2 - 1)^2 \theta^2}{\{ n(c^2 - 1) + c^2 \}^2} + \frac{(c^2 - 1) \theta^2}{\{ n(c^2 - 1) + c^2 \}^2} + \frac{(c^2 - 1)^2 \theta^2}{\{ n(c^2 - 1) + c^2 \}^2} \\ &= \frac{(c^2 - 1) \{ n(c^2 - 1) + c^2 \} \theta^2}{\{ n(c^2 - 1) + c^2 \}^2} \\ &= \frac{(c^2 - 1) \theta^2}{n(c^2 - 1) + c^2}. \end{aligned}$$

Since the estimator in Exercise 7.50 (b) is unbiased, its mean square error is equal to its variance. We have

$$\begin{aligned}\operatorname{var}\{a\overline{X}_n+(1-a)(cS_n)\} &= a^2\operatorname{var}(\overline{X}_n)+(1-a)^2\operatorname{var}(cS_n)\\ &= \frac{n(c^2-1)^2\theta^2}{\{n(c^2-1)+1\}^2}+\frac{(c^2-1)\theta^2}{\{n(c^2-1)+1\}^2}\\ &= \frac{(c^2-1)\theta^2}{n(c^2-1)+1}.\end{aligned}$$

Thus, we have shown that the mean square error of T^* is smaller than the estimator obtained in Exercise 7.50 (b) since

$$\frac{(c^2-1)\theta^2}{n(c^2-1)+c^2}<\frac{(c^2-1)\theta^2}{n(c^2-1)+1}.$$

Solution (c).

Since T^* may be negative then it makes no sense to estimate the parameter $\theta>0$ by a negative value. Thus, we define a new estimator $T^{*^+}=\max(0,T^*)$ which is a more reasonable estimator than T^*. The cumulative distribution of T^{*^+} is

$$\Pr(T^{*^+}\le t)=\Pr\{\max(0,T^*)\le t\}=\Pr(T^*\le t).$$

One may observe that for $t=0$, one has $\Pr(T^{*^+}=0)=\Pr(T^*\le 0)$ which implies that T^{*^+} has non-zero probability at point 0. For $t>0$, one has $\Pr(T^{*^+}\le t)=\Pr(T^*\le t)$. Therefore, T^{*^+} is a random variable formed by a point mass and the positive part of T^*.

Now, suppose the probability density of T^* is f_{T^*}. We show that the mean square error of T^{*^+} is smaller than the mean square error of T^* as follows.

$$\begin{aligned}E_\theta\{(T^{*^+}-\theta)^2\} &= \theta^2\Pr(T^{*^+}=0)+\int_0^\infty (t^*-\theta)^2 f_{T^*}(t^*)dt^*\\ &= \theta^2\Pr(T^*\le 0)+\int_0^\infty (t^*-\theta)^2 f_{T^*}(t^*)dt^*\\ &= \int_{-\infty}^0 \theta^2 f_{T^*}(t^*)dt^*+\int_0^\infty (t^*-\theta)^2 f_{T^*}(t^*)dt^*\\ &< \int_{-\infty}^0 (t^*-\theta)^2 f_{T^*}(t^*)dt^*+\int_0^\infty (t^*-\theta)^2 f_{T^*}(t^*)dt^*\\ &= E_\theta\{(T^*-\theta)^2\}.\end{aligned}$$

Solution (d).

We define the probability density function for standard normal distribution ($N(0,1)$) as

$$\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

Thus, the probability density function for $N(\theta,\theta^2)$ can be written as

$$f_\theta(x)=\frac{1}{\sqrt{2\pi\theta^2}}\exp\left\{-\frac{(x-\theta)^2}{2\theta^2}\right\}=\frac{1}{\theta}\phi\left(\frac{x-\theta}{\theta}\right).$$

The parameter θ is not a location parameter since it cannot be 0. In addition, the parameter θ is also not a scale parameter since for $\theta=1$, one has

$$f_1(x)=\phi(x-1)\neq\phi(x).$$

Hence we have shown that θ is neither location nor scale parameter.

However, if one regards the standard probability density function as

$$\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-1)^2}.$$

Then, the parameter θ can be classified as a scale parameter since

$$f_\theta(x)=\frac{1}{\sqrt{2\pi\theta^2}}\exp\left\{-\frac{(x-\theta)^2}{2\theta^2}\right\}=\frac{1}{\theta}\varphi\left(\frac{x}{\theta}\right).$$

Appendix 1. R codes for the Weilbull model with competing risks

R codes

```
### data generation ###

n = 300

Lambda.true = 1
Mu.true        = 2
Gamma.true    = 3

set.seed(10)
X = rweibull(n,Gamma.true,Lambda.true^(1/Gamma.true))
Y = rweibull(n,Gamma.true,Mu.true^(1/Gamma.true))

Z = pmin(X,Y)
W = rep(1,n)*(X == Z)

### Fixed-point iteration

epsilon = 1e-5
count = 0
Gamma_old = 1
repeat{

   Gamma_new = (sum(Z^Gamma_old*log(Z))/sum(Z^Gamma_old)-sum(log(Z))/n)^-1
   count = count +1
   cat("count = ",count,"Gamma = ",Gamma_new,"¥n")

   if (abs(Gamma_new-Gamma_old) < epsilon) {break}

   Gamma_old = Gamma_new

}
Lambda = (sum(W)/sum(Z^Gamma_new))^(-1/Gamma_new);Lambda
Mu         = ((n-sum(W))/sum(Z^Gamma_new))^(-1/Gamma_new);Mu
Gamma    = Gamma_new ;Gamma
```