Exercise 6.3 [P300]
Let $X_1, \ldots, X_n$ be a random sample from the pdf
\[
f(x|\mu, \sigma) = \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}}, \quad \mu < x < \infty, 0 < \sigma < \infty.
\]
Find a two-dimensional sufficient statistic for $(\mu, \sigma)$.

**Sol:**

The joint pdf of the sample $\mathbf{X}$ is
\[
f(\mathbf{x}|\mu, \sigma) = \prod_{i=1}^{n} \frac{1}{\sigma} \exp\left(-\frac{x_i-\mu}{\sigma}\right), \quad \mu < x_i < \infty, 0 < \sigma < \infty, i = 1, 2, \ldots, n
\]

We can use the indicator function. $I_A(x)$ is the indicator function of the set $A$; that is, it is equal to 1 if $x \in A$ and equal to 0 otherwise. Now $\mu < x_i < \infty$, $i = 1, 2, \ldots, n$ if and only if $\min\{x_1, x_2, \ldots, x_n\} > \mu$.

So we have
\[
f(\mathbf{x}|\mu, \sigma) = \prod_{i=1}^{n} \frac{1}{\sigma} \exp\left(-\frac{x_i-\mu}{\sigma}\right) I(\mu, \infty)(x_i)
\]
\[
= \frac{1}{\sigma^n} \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma}\right) I(-\infty, x_{(1)})(\mu)
\]
\[
= \frac{1}{\sigma^n} \exp\left(\frac{n\mu}{\sigma}\right) \exp\left(-\frac{\sum_{i=1}^{n}x_i}{\sigma}\right) I(-\infty, x_{(1)})(\mu)
\]

Since $f(x|\mu, \sigma) = h(x)g(T(\mathbf{X})|\mu, \sigma)$, where $h(x) = 1$ and
\[
g(T(\mathbf{X})|\mu, \sigma) = \frac{1}{\sigma^n} \exp\left(\frac{n\mu}{\sigma}\right) \exp\left(-\frac{\sum_{i=1}^{n}x_i}{\sigma}\right) I(-\infty, x_{(1)})(\mu), T_1(\mathbf{X}) = x_{(1)}, T_2(\mathbf{X}) = \sum_{i=1}^{n} X_i
\]
Therefore, by the Factorization Theorem, $(x_{(1)}, \sum_{i=1}^{n} X_i)$ is a two-dimensional sufficient statistic for $(\mu, \sigma)$. 


Exercise 6.5 [P300]
Let $X_1, \ldots, X_n$ be independent random variables with pdfs
\[
f(x_i|\theta) = \begin{cases} \frac{1}{2\theta} & -i(\theta - 1) < x_i < i(\theta + 1) \\ 0 & \text{otherwise}, \end{cases}
\]
where $\theta > 0$. Find a two-dimensional sufficient statistic for $\theta$.

Sol :

The joint pdf of the sample $X$ is
\[
f(x|\theta) = \prod_{i=1}^{n} \frac{1}{2\theta} I_{(-i(\theta-1),i(\theta+1))}(x_i)
\]
Now $-i(\theta - 1) < x_i < i(\theta + 1)$, using the indicator function, we can see the inequality is $-i(\theta - 1) < \min x_i$ and $\max x_i < i(\theta + 1)$.

\[
\Rightarrow \left\{ \begin{array}{ll}
-i(\theta - 1) < \min x_i & \Rightarrow \left\{ \begin{array}{l}
\theta - 1 > -\min \frac{x_i}{i}
\end{array} \right. \\
i(\theta + 1) > \max x_i & \Rightarrow \left\{ \begin{array}{l}
\theta > 1 - \min \frac{x_i}{i}
\end{array} \right.
\end{array} \right.
\]

So we have
\[
f(x|\theta) = \prod_{i=1}^{n} \frac{1}{2\theta} I_{(-i(\theta-1),i(\theta+1))}(x_i)
\]
Since $f(x|\mu, \sigma) = h(x)g(T(X)|\mu, \sigma)$, where $h(x) = \prod_{i=1}^{n} \frac{1}{i}$ and $g(T(X)|\theta) = 
\]
\[
\left( \frac{1}{2\theta} \right)^n I_{\left(1-\min \frac{x_i}{i}, \infty\right)}(\theta) I_{\left(\max \frac{x_i}{i}, 1-\infty\right)}(\theta)
\]
Therefore, by the Factorization Theorem, $(\min \frac{X_i}{i}, \max \frac{X_i}{i})$ is a two-dimensional sufficient statistic for $\theta$. 

Exercise 6.6 [P300]
Let $X_1,\ldots,X_n$ be a random sample from a gamma($\alpha, \beta$) population. Find a two-dimensional sufficient statistic for ($\alpha, \beta$).

Sol:

The joint pdf of the sample $X$ is
\[
f(x|\alpha, \beta) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} \exp\left(-\frac{x_i}{\beta}\right)
\]
\[
= \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \exp\left(-\frac{\sum_{i=1}^{n} x_i}{\beta}\right)
\]

Since $f(x|\alpha, \beta) = h(x)g(T(X)|\alpha, \beta)$, where $h(x) = 1$ and $g(T(X)|\alpha, \beta) = \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \exp\left(-\frac{\sum_{i=1}^{n} x_i}{\beta}\right)$, $T_1(X) = \prod_{i=1}^{n} X_i$, $T_2(X) = \sum_{i=1}^{n} X_i$

Therefore, by the Factorization Theorem, $(\prod_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i)$ is a two-dimensional sufficient statistic for ($\alpha, \beta$).
Exercise 6.7 [P300]
Let \( f(x, y|\theta_1, \theta_2, \theta_3, \theta_4) \) be the bivariate pdf for the uniform distribution on the rectangle with lower left corner \((\theta_1, \theta_2)\) and upper right corner \((\theta_3, \theta_4)\) in \(\mathbb{R}^2\). The parameters satisfy \( \theta_1 < \theta_3 \) and \( \theta_2 < \theta_4 \). Let \((X_1, Y_1), \ldots, (X_n, Y_n)\) be a random sample from this pdf. Find a four-dimensional sufficient statistic for \((\theta_1, \theta_2, \theta_3, \theta_4)\).

Sol :

The joint pdf is
\[
f(x, y|\theta) = \prod_{i=1}^{n} \frac{1}{\theta_3 - \theta_1} \frac{1}{\theta_4 - \theta_2} I_{(\theta_1, \theta_3)}(x_i) I_{(\theta_2, \theta_4)}(y_i)
\]
Using the indicator function,
\[
\theta_1 \leq \min\{x_1, \ldots, x_n\} = x_{(1)}, \max\{x_1, \ldots, x_n\} = x_{(n)} \leq \theta_3,
\]
\[
\theta_2 \leq \min\{y_1, \ldots, y_n\} = y_{(1)}, \max\{y_1, \ldots, y_n\} = y_{(n)} \leq \theta_4
\]
So we have
\[
f(x, y|\theta) = \prod_{i=1}^{n} \frac{1}{\theta_3 - \theta_1} \frac{1}{\theta_4 - \theta_2} I_{(-\infty, x_{(1)})}(\theta_1) I_{(-\infty, x_{(n)})}(\theta_2) I_{(x_{(n)}, \infty)}(\theta_3) I_{(y_{(n)}, \infty)}(\theta_4)
\]
Since \( f(x, y|\theta) = h(x)g(T(X)|\theta) \), where \( h(x) = 1 \) and \( g(T(X)|\theta) = \)
\[
\left(\frac{1}{\theta_3 - \theta_1}\right)^n \left(\frac{1}{\theta_4 - \theta_2}\right)^n I_{(-\infty, x_{(1)})}(\theta_1) I_{(-\infty, y_{(1)})}(\theta_2) I_{(x_{(n)}, \infty)}(\theta_3) I_{(y_{(n)}, \infty)}(\theta_4)
\]
\( T_1(X) = X_{(1)}, \ T_2(X) = Y_{(1)}, \ T_3(X) = X_{(n)}, \ T_4(X) = Y_{(n)}. \)
Therefore, by the Factorization Theorem, \((X_{(1)}, Y_{(1)}, X_{(n)}, Y_{(n)})\) is a four-dimensional sufficient statistic for \((\theta_1, \theta_2, \theta_3, \theta_4)\).
Exercise 6.8 [P301]

Let $X_1, \ldots, X_n$ be a random sample from a population with location pdf $f(x - \theta)$. Show that the order statistics, $T(X_1, \ldots, X_n) = (X_{(1)}, \ldots, X_{(n)})$, are a sufficient statistic for $\theta$ and no further reduction is possible.

Sol :

We know $X_1, \ldots, X_n \sim f(x - \theta), \theta \in \mathbb{R}$.

The joint pdf is

$$f(x|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} f(x_i - \theta) = \prod_{i=1}^{n} f(x_{(i)} - \theta)$$

Where $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ are the order statistics of $X_1, \ldots, X_n$.

Since $f(x, y|\theta) = h(x)g(T(X)|\theta)$, where $h(x) = 1$ and $g(T(X)|\theta) = \prod_{i=1}^{n} f(x_{(i)} - \theta)$, $T_i(X) = X_{(i)}$, $i = 1, \ldots, n$

Therefore, by the Factorization Theorem, $(X_{(1)} , \ldots, X_{(n)})$ are a sufficient statistic for $\theta$.

That no further reduction is possible without further restrictions on $f$. In this case, it suffices to notice the ratio

$$\frac{\prod_{i=1}^{n} f(x_{(i)} - \theta)}{\prod_{i=1}^{n} f(y_{(i)} - \theta)}$$

is general independent of $\theta$ only when $T(X) = T(Y)$. 