Quiz#2, Mathematical Statistics II, 2014 Spring

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NOTE: z_α represents the upper α-percentile of $N(0,1)$, or $z_\alpha = \Phi^{-1}(1-\alpha)$.

You can use similar notations, such as $t_{n,\alpha}$ and $\chi^2_{n,\alpha}$ for t-distribution and χ^2-distribution, respectively.

+2/2

1. Let $X_1,\dots,X_n \overset{iid}{\sim} N(\mu,\sigma^2)$, where μ is known. Derive a size α MP test for $H_0:\sigma^2=\sigma_0^2$ vs. $H_1:\sigma^2=\sigma_1^2$, where $\sigma_0^2>\sigma_1^2$.

by N-P lemma.

$$\frac{f(x\mid\mu,\sigma_1^2)}{f(x\mid\mu,\sigma_0^2)} = \frac{\left(\frac{1}{\sqrt{2\pi}\sigma_1}\right)^n \exp\left(-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma_1^2}\right)}{\left(\frac{1}{\sqrt{2\pi}\sigma_0}\right)^n \exp\left(-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma_0^2}\right)} = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left(-\sum_{i=1}^n (x_i-\mu)^2\left[\frac{1}{2\sigma_1^2}-\frac{1}{2\sigma_0^2}\right]\right) >$$

$$\Leftrightarrow \exp\left(-\sum_{i=1}^n (x_i-\mu)^2\left[\frac{1}{2\sigma_1^2}-\frac{1}{2\sigma_0^2}\right]\right) > k\cdot\left(\frac{\sigma_1}{\sigma_0}\right)^n.$$

$$\Leftrightarrow \sum_{i=1}^n (x_i-\mu)^2\left[\frac{1}{2\sigma_0^2}-\frac{1}{2\sigma_1^2}\right] > \log k\left(\frac{\sigma_1}{\sigma_0}\right)^n \quad \left(\because \sigma_0^2>\sigma_1^2 \Leftrightarrow \frac{1}{2\sigma_0^2}<\frac{1}{2\sigma_1^2}\right)$$

$$\Leftrightarrow \sum_{i=1}^n (x_i-\mu)^2 < \left(\frac{1}{2\sigma_0^2}-\frac{1}{2\sigma_1^2}\right)\log k\left(\frac{\sigma_1}{\sigma_0}\right)^n$$

$$\Leftrightarrow \frac{\sum_{i=1}^n (x_i-\mu)^2}{\sigma_0^2} < k'$$

$\therefore$ rejection region is $\left\{x \mid \frac{\sum_{i=1}^n (x_i-\mu)^2}{\sigma_0^2} < k'\right\}$.

with size α

$$\Rightarrow \alpha = P_{\sigma_0}\left(\frac{\sum_{i=1}^n (x_i-\mu)^2}{\sigma_0^2} < k'\right) \longrightarrow \quad \because \frac{x_i-\mu}{\sigma_0}\sim N(0,1)$$

$$\Rightarrow \left(\frac{x_i-\mu}{\sigma_0}\right)^2 \sim \chi^2_1$$

$$\Rightarrow \sum_{i=1}^n \left(\frac{x_i-\mu}{\sigma_0}\right)^2 \sim \chi^2_n$$

$\Rightarrow k' = \chi^2_{n,1-\alpha}$ with

$\chi^2_{n,1-\alpha}$

$$\Rightarrow \delta(x) = \begin{cases} 1 & \text{if } \frac{\sum_{i=1}^n (x_i-\mu)^2}{\sigma_0^2} < \chi^2_{n,1-\alpha} \\ 0 & \text{o.w.} \end{cases}$$

is size α MP-test

+3/3

2 Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, where σ^2 is known.

Let $\lambda(x)$ be the likelihood ratio statistics for $H_0: \mu \le \mu_0$ vs. $H_1: \mu > \mu_0$. The test rejects H_0 if $\lambda(x) < c$ for some $0 < c < 1$.

1) Rewrite $\lambda(x) < c$ and simplify the inequality (using c; not only the answer).

2) Determine the value of c to be a size α test.

3) Draw the power curve $\beta(\mu)$ in the range of $\mu > \mu_0$ (require details).

1. to test $H_0: \mu \le \mu_0$ v.s. $H_1: \mu > \mu_0$, σ^2: known.

$$\Rightarrow \lambda(x) = \frac{\sup_{\mu \in \Theta_0} L(\mu \mid \underline{x})}{\sup_{\mu \in \Theta} L(\mu \mid \underline{x})} = \frac{\exp\left(-\frac{\sum_{i=1}^n (x_i - \hat{\mu})^2}{2\sigma^2}\right)}{\exp\left(-\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{2\sigma^2}\right)}$$

Note: In $\mu \in \Theta$ $\Rightarrow \hat{\mu} = \bar{x}$ is MLE of μ. In $\mu \in \Theta_0$.

$$\hat{\mu} = \begin{cases} \bar{x} & \text{if } \bar{x} \le \mu_0 \\ \mu_0 & \text{o.w.} \end{cases}$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \hat{\mu})^2 - \sum_{i=1}^n (x_i - \bar{x})^2\right)\right) < c$$

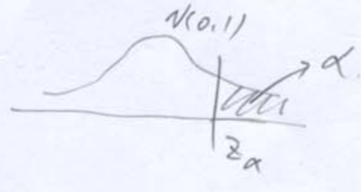
$\Leftrightarrow$ $\exp\left(-\frac{1}{2\sigma^2}(n(\bar{x} - \hat{\mu})^2)\right) < c$ and $\bar{x} > \mu_0$ * $\left(\because \bar{x} \le \mu_0 \Rightarrow \lambda(x) = 1 \Rightarrow \text{just consider } \bar{x} > \mu_0 \right)$

2. $\Leftrightarrow$ $\frac{n(\bar{x} - \hat{\mu})^2}{\sigma^2} > -2\log c$ and $\bar{x} > \mu_0$

$\Leftrightarrow$ $\frac{n(\bar{x} - \mu_0)^2}{\sigma^2} > -2\log c$ and $\bar{x} > \mu_0$ $(\bar{x} - \mu_0 > 0)$.

$\Leftrightarrow$ $\frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma} > \sqrt{-2\log c}$ Note: $\frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma} \sim N(0,1)$.

with size α $\Rightarrow$ let $\sqrt{-2\log c} = Z_\alpha$,



$\Rightarrow \sqrt{-2\log c} = Z_\alpha$

$-2\log c = Z_\alpha^2$

$c = \exp\left(-\frac{Z_\alpha^2}{2}\right)$ *

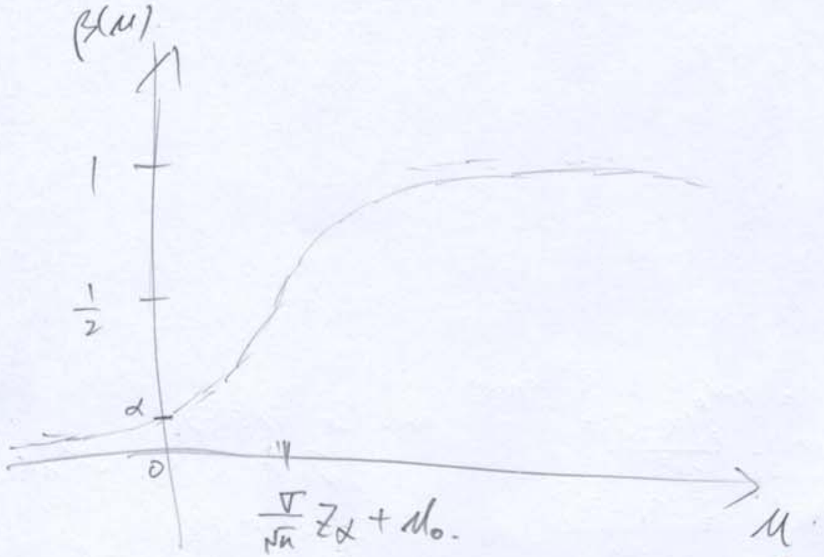
3. $\beta(\mu) = P_\mu\left(\frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma} > Z_\alpha\right) = P_\mu\left(\frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} + \frac{\sqrt{n}(\mu - \mu_0)}{\sigma} > Z_\alpha\right)$

$= P_\mu\left(\frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} > -\frac{\sqrt{n}(\mu - \mu_0)}{\sigma} + Z_\alpha\right) = 1 - \Phi\left(Z_\alpha - \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}\right)$

$$\Rightarrow \beta(\mu) = 1 - \Phi\left(Z_\alpha - \frac{\sqrt{n}(\mu - \mu_0)}{\sigma} \right)$$

$$Z_\alpha = \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}$$

$$\Rightarrow \frac{\sigma}{\sqrt{n}} Z_\alpha + \mu_0 = \mu$$



$$\lim_{\mu \to \infty} \beta(\mu) = 1 - \Phi(-\infty) = 1$$

$$\lim_{\mu \to -\infty} \beta(\mu) = 1 - 1 = 0$$

+2/3

3. Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{exponential}(\lambda)$, where $\lambda = EX_1$. Let $T = \sum_{i=1}^{n} X_i$ be a sufficient statistics for λ

i) Show that $Q(T,\lambda) = 2T/\lambda$ is a pivotal quantity for λ.

ii) Derive the pdf of $Q(T,\lambda)$.

iii) What is the distribution of $Q(T,\lambda)$?

i). $X_1 \cdots X_n \sim exp(\lambda)$.

$f(x) = (\frac{1}{\lambda})^n exp(-\frac{\sum_{i=1}^n x_i}{\lambda})$ $T = \sum_{i=1}^{n} x_i \sim$ Gamma (n, λ).

$\Rightarrow f_T(t) = \frac{\lambda^{-n}}{\Gamma(n)} t^{n-1} exp(-\frac{t}{\lambda})$, p.d.f. of T

let $Q(T,\lambda) = \frac{2T}{\lambda}$ $\Rightarrow$. $T = \frac{\lambda}{2} Q(T,\lambda)$ $\Rightarrow J = \frac{\lambda}{2}$

$\Rightarrow g_{Q(T,\lambda)}(k) = f_T(\frac{\lambda k}{2}) \cdot |J| = \frac{\lambda^{-n}}{\Gamma(n)} (\frac{\lambda k}{2})^{n-1} exp(-\frac{1}{\lambda} \cdot \frac{k\lambda}{2}) \cdot \frac{\lambda}{2}$

$= \frac{2^{-n}}{\Gamma(n)} exp(-\frac{k}{2}) K^{n-1}$

$\Rightarrow Q(T,\lambda) = \frac{2T}{\lambda}$ the p.d.f is $g_{Q(T,\lambda)}(K) = \frac{2^{-n}}{\Gamma(n)} K^{n-1} exp(-\frac{K}{2})$, $K>0$

does have λ.

$\Rightarrow Q(T,\lambda)$ is a pivotal quantity for λ.

(ii) and the p.d.f is $f_{Q(T,\lambda)}(K) = \frac{2^{-n}}{\Gamma(n)} K^{n-1} exp(-\frac{K}{2})$ $K>0$.

(iii)

$Q(T,\lambda) \sim$ Gamma $(n, 2)$. ×