2) By 1). $\widehat{\beta\tau} = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$ is the MLE of $\beta\tau$.

$E(\bar{Y}) = E(Y_1) = \tau\beta$; so, $\widehat{\beta\tau}$ is unbiased ✓

($\because Y_1, \dots, Y_n \overset{iid}{\sim} P_o(\tau\beta)$ $\therefore E(Y_i) = \tau\beta$, $i=1,\dots,n$)

Quiz#1, Mathematical Statistics II, 2014 Spring

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Name: Liao Yen Meng

+4/4

1. [+4] Let $X_1,\dots,X_n \overset{iid}{\sim} \text{Poisson}(\tau)$, $Y_1,\dots,Y_n \overset{iid}{\sim} \text{Poisson}(\tau\beta)$, where $X_i \perp Y_i$, $i = 1,\dots,n$. Let $\theta = (\beta, \tau) \in (0,\infty)\times(0,\infty)$.

   1) [+2] Derive the MLE (without checking the negative definiteness of Hessian)

   $\boxed{2)}$ [+2] Derive the MLE of $\beta\tau$. Is this unbiased?

1). $L(\tau,\beta|\underline{x},\underline{y}) = \left[\prod_{i=1}^{n} f(x_i;\tau)\right]\left[\prod_{i=1}^{n} f(y_i;\tau,\beta)\right] = \frac{1}{\prod_{i=1}^{n} x_i! \prod_{i=1}^{n} y_i!} e^{-n\tau} \tau^{\sum_{i=1}^{n} x_i} e^{-n\tau\beta} (\tau\beta)^{\sum_{i=1}^{n} y_i}$

$= \frac{1}{\prod_{i=1}^{n} x_i! \prod_{i=1}^{n} y_i!} \exp(-n\tau(1+\beta))\, \tau^{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i} \beta^{\sum_{i=1}^{n} y_i}$; $\ln L(\tau,\beta|\underline{x},\underline{y}) = \ln\left(\frac{1}{\prod_{i=1}^{n} x_i! \prod_{i=1}^{n} y_i!}\right) - n\tau(1+\beta) + \left(\sum_{i=1}^{n}(x_i+y_i)\right)\ln\tau + \sum_{i=1}^{n} y_i \ln\beta$.

$$\begin{cases} \frac{\partial}{\partial\tau}\ln L(\tau,\beta|\underline{x},\underline{y}) = -n(1+\beta) + \frac{\sum_{i=1}^{n}(x_i+y_i)}{\tau} = 0 & \text{---} ① \\ \frac{\partial}{\partial\beta}\ln L(\tau,\beta|\underline{x},\underline{y}) = -n\tau + \frac{\sum_{i=1}^{n} y_i}{\beta} = 0 & \text{---} ② \end{cases}$$

$$\Rightarrow \begin{cases} \hat{\tau} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x} \\ \hat{\beta} = \frac{\frac{1}{n}\sum_{i=1}^{n} y_i}{\frac{1}{n}\sum_{i=1}^{n} x_i} = \frac{\bar{y}}{\bar{x}} \end{cases}$$ ✓ is the MLE of τ, β.

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2. [+3] Let $X_1,\dots,X_n \overset{iid}{\sim} N(\theta, 1)$, where $\theta \in [a,b]$ for some $a < b$.

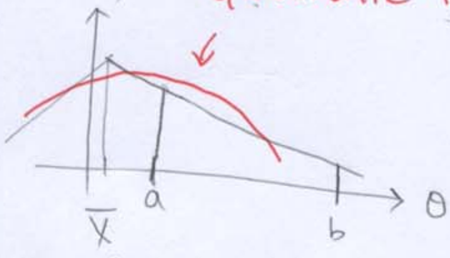
   Derive the MLE $\hat{\theta}$. Also, draw Figures to show that $\hat{\theta}$ maximizes the likelihood function [3 cases need to be drawn].

$L(\theta|\underline{x}) = \prod_{i=1}^{n} f(x_i;\theta) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2\right)$, $\ln L(\theta|\underline{x}) = -\frac{n}{2}\ln 2\pi - \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2$

$\frac{d}{d\theta}\ln L(\theta|\underline{x}) = \sum_{i=1}^{n}(x_i-\theta) = 0 \Rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$, and $\frac{d^2}{d\theta^2}\ln L(\theta|\underline{x})\big|_{\theta=\hat{\theta}} < 0$

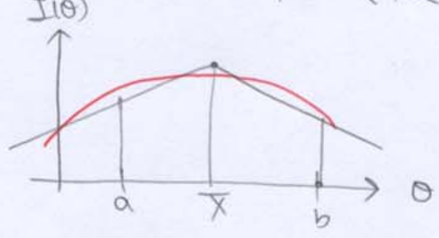
✓ <case 1> if $\bar{x} < a$.

quadratic function



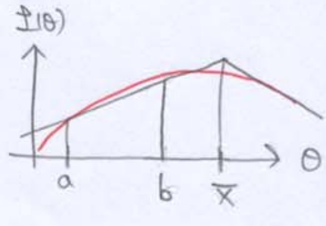
then $\hat{\theta} = a$ is mle of θ.

✓ <case 2> if, $a \le \bar{x} \le b$



then $\hat{\theta} = \bar{x}$ is mle of θ

✓ <case 3> if $\bar{x} > b$



then $\hat{\theta} = b$ is mle of θ

by <case 1, 2, 3>, $\hat{\theta} = \begin{cases} a ; & \text{if } \bar{x} < a \\ \bar{x} ; & \text{if } a \le \bar{x}_n \le b \\ b ; & \text{if } b < \bar{x}_n \end{cases}$ is mle of θ. ✓

3. [+4] Let $X_1, \ldots, X_n \overset{iid}{\sim} f(x|\eta)$, where

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$$f(x|\eta) = \begin{cases} \exp(\eta x - \varphi(\eta)) & \text{if} \quad x \le 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$$

, where $\eta > 0$ is unknown.

1) [+2] Derive the form of $\varphi(\eta)$

2) [+2] Derive the MLE of η

1). $1 = \int_{-\infty}^{1} \exp(\eta x - \phi(\eta))\, dx = \int_{-\infty}^{1} e^{-\phi(\eta)} e^{\eta x}\, dx = e^{-\phi(\eta)} \int_{-\infty}^{1} e^{\eta x}\, dx$

$\Rightarrow e^{\phi(\eta)} = \frac{1}{\eta}\int_{-\infty}^{1} e^{\eta x}\, d(x \cdot \eta) = \frac{1}{\eta}\left(e^{\eta x}\Big|_{x=-\infty}^{1}\right) = \frac{1}{\eta}\left(e^{\eta}\right) = \frac{e^{\eta}}{\eta}$

$\Rightarrow \phi(\eta) = \ln \frac{e^{\eta}}{\eta} = \ln e^{\eta} - \ln \eta = \eta - \ln \eta$ # ✓

2). $f(x|\eta) = \exp\{\eta x - \eta + \ln \eta\} = \eta e^{\eta(x-1)}$; $x \le 1$ ✓

$L(\eta|\underline{X}) = \prod_{i=1}^{n} f(x_i|\eta) = \eta^n \exp\left(\eta \sum_{i=1}^{n}(X_i - 1)\right)$, $\ln L(\eta|\underline{X}) = n \ln \eta + \eta \sum_{i=1}^{n}(X_i - 1)$

$\frac{d}{d\eta} \ln L(\eta|\underline{X}) = \frac{n}{\eta} + \sum_{i=1}^{n}(X_i - 1) = 0 \Rightarrow \hat{\eta} = \frac{-n}{\sum_{i=1}^{n}(X_i - 1)}$; and $\frac{d^2}{d\eta^2} \ln L(\eta|\underline{X})\Big|_{\eta=\hat{\eta}} < 0$

$\therefore \hat{\eta} = \frac{-n}{\sum_{i=1}^{n}(X_i - 1)}$ is the mle of η #.

✓

(-1)

$\overset{\text{"}}{?}$

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4. [+7]Let $L(\theta)$ be a likelihood function for $\theta \in R$. Let $\eta = \tau(\theta)$ be a one-to-one transformation and $\tau^{-1}(\eta) = \theta$ be its inverse. Assume that the first and second derivatives $L'(\theta)$, $L''(\theta) < 0$, $\tau'(\theta) \neq 0$ and $\tau''(\theta)$ exist.

1) [+1]Define the likelihood function $L^{*}(\eta)$ for η.
2) [+2]Find the first derivative $L^{*'}(\eta)$, as a function of η.
3) [+2]Find the second derivative $L^{*''}(\eta)$, as a function of η.
4) [+2]State the invariance of the MLE. Then, prove the invariance using 2) and 3).

1) $L(\theta) = L(\tau^{-1}(\eta)) = L^{*}(\eta)$ is the likelihood function for η.

2) $\frac{d}{d\eta} L^{*}(\eta) = \frac{d}{d\eta} L(\tau^{-1}(\eta)) = \frac{d}{d\tau^{-1}(\eta)} L(\tau^{-1}(\eta)) \cdot \frac{d\tau^{-1}(\eta)}{d\eta} = L'(\tau^{-1}(\eta)) \cdot \frac{d\tau^{-1}(\eta)}{d\eta}$ — (*)

, where $\frac{d\tau^{-1}(\eta)}{d\eta} = \frac{1}{\tau'(\tau^{-1}(\eta))}$; so (*) : $\frac{d}{d\eta} L^{*}(\eta) = L'(\tau^{-1}(\eta)) \frac{1}{\tau'(\tau^{-1}(\eta))}$

$\left(\because \tau(\tau^{-1}(\eta)) = \eta \xRightarrow{\frac{d}{d\eta}} \tau'(\tau^{-1}(\eta)) \cdot (\tau^{-1})'(\eta) = 1 \right.$
$\left. \Rightarrow (\tau^{-1})'(\eta) = \frac{1}{\tau'(\tau^{-1}(\eta))} \text{ and } \tau'(\tau^{-1}(\eta)) \neq 0, \forall \eta \right)$

3) $\frac{d^2}{d\eta^2} L^{*}(\eta) = \frac{dL'(\tau^{-1}(\eta))}{d\eta} \left(\frac{1}{\tau'(\tau^{-1}(\eta))} \right) + L'(\tau^{-1}(\eta)) \left(\frac{-\frac{d\tau'(\tau^{-1}(\eta))}{d\eta}}{[\tau'(\tau^{-1}(\eta))]^2} \right)$ — (*)

; where $\frac{dL'(\tau^{-1}(\eta))}{d\eta} = \frac{dL'(\tau^{-1}(\eta))}{d\tau^{-1}(\eta)} \frac{d\tau^{-1}(\eta)}{d\eta} = L''(\tau^{-1}(\eta)) \frac{1}{\tau'(\tau^{-1}(\eta))}$

and $\frac{d\tau'(\tau^{-1}(\eta))}{d\eta} = \frac{d\tau'(\tau^{-1}(\eta))}{d\tau^{-1}(\eta)} \frac{d\tau^{-1}(\eta)}{d\eta} = \tau''(\tau^{-1}(\eta)) \frac{1}{\tau'(\tau^{-1}(\eta))}$

so (*) : $\frac{d^2}{d\eta^2} L^{*}(\eta) = L''(\tau^{-1}(\eta)) \frac{1}{[\tau'(\tau^{-1}(\eta))]^2} - L'(\tau^{-1}(\eta)) \left(\frac{\tau''(\tau^{-1}(\eta))}{[\tau'(\tau^{-1}(\eta))]^3} \right)$

4) invariance of MLE :

$L(\theta|x)$ is the likelihood function of θ & $\hat{\theta}$ is the MLE of θ

if $\tau(\theta) = \eta$ with τ^{-1} exist, then $\hat{\eta} = \tau(\hat{\theta})$ is the MLE of η

(next page)

$\because \hat{\theta}$ is MLE of θ, so $\mathcal{L}(\hat{\theta}|\underline{x}) \geq \mathcal{L}(\theta|\underline{x})$, $\forall \theta$.

that is, $\frac{d}{d\theta}\mathcal{L}(\theta|\underline{x})\Big|_{\theta=\hat{\theta}} = 0$ & $\frac{d^2}{d\theta^2}\mathcal{L}(\theta|\underline{x})\Big|_{\theta=\hat{\theta}} < 0$.

by 2). $(\mathcal{L}^*)'(\eta) = \mathcal{L}'(\tau^{-1}(\eta))\frac{1}{\tau'(\tau^{-1}(\eta))} \Rightarrow (\mathcal{L}^*)'(\eta)\Big|_{\eta=\hat{\eta}} = \mathcal{L}'(\tau^{-1}(\tau(\hat{\theta})))\frac{1}{\tau'(\tau^{-1}(\tau(\hat{\theta})))}$

$= \mathcal{L}'(\hat{\theta})\frac{1}{\tau'(\hat{\theta})} = 0$

$\because)\ \tau'(\hat{\theta}) \neq 0.$

by 3), $(\mathcal{L}^*)''(\eta) = \mathcal{L}''(\tau^{-1}(\eta))\frac{1}{[\tau'(\tau^{-1}(\eta))]^2} - \mathcal{L}'(\tau^{-1}(\eta))\left(\frac{\tau''(\tau^{-1}(\eta))}{[\tau'(\tau^{-1}(\eta))]^3}\right)$

$\Rightarrow (\mathcal{L}^*)''(\eta)\Big|_{\eta=\hat{\eta}} = \mathcal{L}''(\tau^{-1}(\hat{\eta}))\frac{1}{[\tau'(\tau^{-1}(\hat{\eta}))]^2} - \mathcal{L}'(\tau^{-1}(\hat{\eta}))\left(\frac{\tau''(\tau^{-1}(\hat{\eta}))}{[\tau'(\tau^{-1}(\hat{\eta}))]^3}\right)$

$= \mathcal{L}''(\tau^{-1}(\tau(\hat{\theta})))\frac{1}{[\tau'(\tau^{-1}(\tau(\hat{\theta})))]^2} - \mathcal{L}'(\tau^{-1}(\tau(\hat{\theta})))\left(\frac{\tau''(\tau^{-1}(\tau(\hat{\theta})))}{[\tau'(\tau^{-1}(\tau(\hat{\theta})))]^3}\right)$

$= \mathcal{L}''(\hat{\theta})\frac{1}{[\tau'(\hat{\theta})]^2} - \mathcal{L}'(\hat{\theta})\frac{\tau''(\hat{\theta})}{[\tau'(\hat{\theta})]^3} = \frac{1}{[\tau'(\hat{\theta})]^2}\mathcal{L}''(\hat{\theta}) < 0.$

; so, $\hat{\eta}$ is the MLE of η. *.