Midterm Exam 5/2(Fri), Mathematical Statistics II, 2014 Spring

5 questions (please check), Total score = 35 points

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$33/35$

+6 1. [+6] Let $X_1,\ldots,X_n \overset{iid}{\sim} f(x|\eta,\tau)$, where

$$f(x\mid\eta,\tau)=\begin{cases}\exp\{\eta x-\varphi(\eta,\tau)\} & \text{if} \quad x\le\tau\\ 0 & \text{if} \quad x>\tau\end{cases}$$

, where both τ and $\eta>0$ are unknown.

1) [+2] Derive the form of $\varphi(\eta,\tau)$.

2) [+4] Derive the MLE of (η,τ).

+2

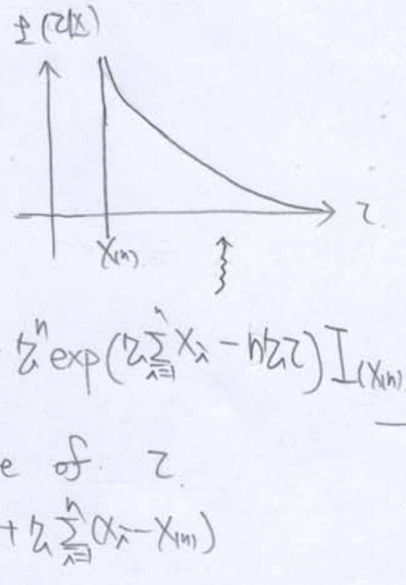
1) $1=\int_{-\infty}^{\tau}\exp\{\eta x-\phi(\eta,\tau)\}\,dx = \exp\{-\phi(\eta,\tau)\}\cdot\int_{-\infty}^{\tau} e^{\eta x}\,dx = e^{-\phi(\eta,\tau)}\cdot\frac{1}{\eta}e^{\eta x}\Big|_{x=-\infty}^{\tau}$

$=e^{-\phi(\eta,\tau)}\frac{1}{\eta}e^{\eta\tau} \Rightarrow e^{-\phi(\eta,\tau)}=\eta e^{-\eta\tau} \Rightarrow -\phi(\eta,\tau)=\ln\eta-\eta\tau$ ✓

$\Rightarrow \phi(\eta,\tau)=\eta\tau-\ln\eta$ # ✓

2) by 1), $f(x|\eta,\tau)=\begin{cases}\eta\exp(\eta(x-\tau)) & ; x\le\tau\\ 0 & ; x>\tau\end{cases}$

+4



$L(\eta,\tau|\underline{X})=\prod_{i=1}^{n}f(x_i|\eta,\tau)=\eta^n\exp\left(\eta\sum_{i=1}^{n}(x_i-\tau)\right)\cdot I_{(X_{(n)},\infty)}(\tau)=\eta^n\exp\left(\eta\sum_{i=1}^{n}X_i-n\eta\tau\right)I_{(X_{(n)},\infty)}(\tau)$ —(*)

fix η ; $\sup_{\tau}L(\tau|\underline{x}) \Leftrightarrow \inf_{\tau}\tau$, so $\hat\tau=X_{(n)}$ is mle of τ.

Then (*) : $L(\eta|\underline{X})=\eta^n\exp\left(\eta\sum_{i=1}^{n}(x_i-X_{(n)})\right)$, $\ln L(\eta|\underline{X})=n\ln\eta+\eta\sum_{i=1}^{n}(x_i-X_{(n)})$

$\frac{d}{d\eta}\ln L(\eta|\underline{X})=\frac{n}{\eta}+\sum_{i=1}^{n}(x_i-X_{(n)})=0 \Rightarrow \hat\eta=\frac{-n}{\sum_{i=1}^{n}(x_i-X_{(n)})}$, and $\frac{d^2}{d\eta^2}\ln L(\eta|\underline{X})\Big|_{\eta=\hat\eta}<0$

$\therefore$ $\hat\eta$ is mle of η.

that is, $\begin{cases}\hat\tau=X_{(n)} \text{ is mle of } \tau\\ \hat\eta=\frac{-n}{\sum_{i=1}^{n}(x_i-X_{(n)})} \text{ is mle of } \eta\end{cases}$ ✓

+8

2. [+8]Let $L(\theta|\mathbf{x})$ be a likelihood function for $\theta \in R$, and $\hat{\theta}$ be the MLE. Let $\eta = \tau(\theta)$ be a one-to-one transformation and $\tau^{-1}(\eta) = \theta$ be its inverse. Assume that $L'(\theta|\mathbf{x})$, $L''(\theta|\mathbf{x}) < 0$, $\tau'(\theta) \neq 0$ and $\tau''(\theta)$ exist.

1) [+2]Find the first derivative $L^{*'}(\eta|\mathbf{x})$ with respect to η, where $L^{*}(\eta|\mathbf{x})$ is the likelihood for η.

2) [+2]Find the second derivative $L^{*''}(\eta|\mathbf{x})$ with respect to η.

3) [+1]Derive the MLE of η by using the invariance.

4) [+3] Check the derivative conditions for the MLE based on 1) and 2).

1) $L(\theta|\underline{x}) = L(\tau^{-1}(\eta)|\underline{x}) = L^{*}(\eta|\underline{x})$

$\frac{d}{d\eta}L^{*}(\eta|\underline{x}) = \frac{d}{d\eta}L(\tau^{-1}(\eta)|\underline{x}) = \frac{d}{d\tau^{-1}(\eta)}L(\tau^{-1}(\eta)|\underline{x}) \cdot \frac{d\tau^{-1}(\eta)}{d\eta}$ —— (*)

, where $\frac{d}{d\tau^{-1}(\eta)}L(\tau^{-1}(\eta)|\underline{x}) = L'(\tau^{-1}(\eta)|\underline{x})$, and $\frac{d\tau^{-1}(\eta)}{d\eta} = \frac{1}{\tau'(\tau^{-1}(\eta))}$

$\left(\begin{array}{l} \because \tau(\tau^{-1}(\eta)) = \eta \\ \Rightarrow \tau'(\tau^{-1}(\eta))(\tau^{-1})'(\eta) = 1 \\ \Rightarrow (\tau^{-1})'(\eta) = \frac{1}{\tau'(\tau^{-1}(\eta))},\ \tau'(\tau^{-1}(\eta)) \neq 0 \end{array}\right)$

so (*) : $L^{*'}(\eta|\underline{x}) = L'(\tau^{-1}(\eta)|\underline{x})\frac{1}{\tau'(\tau^{-1}(\eta))}$ #

2) $\frac{d^2}{d\eta^2}L^{*}(\eta|\underline{x}) = \frac{d}{d\eta}\left(\frac{L'(\tau^{-1}(\eta)|\underline{x})}{\tau'(\tau^{-1}(\eta))}\right) = \frac{1}{[\tau'(\tau^{-1}(\eta))]^2}\left(\frac{d}{d\eta}L'(\tau^{-1}(\eta)|\underline{x})\cdot\tau'(\tau^{-1}(\eta)) - L'(\tau^{-1}(\eta)|\underline{x})\frac{d}{d\eta}\tau'(\tau^{-1}(\eta))\right)$ —— (**)

, where $\frac{d}{d\eta}L'(\tau^{-1}(\eta)|\underline{x}) = \frac{d}{d\tau^{-1}(\eta)}L'(\tau^{-1}(\eta)|\underline{x})\cdot\frac{d\tau^{-1}(\eta)}{d\eta}$

$= L''(\tau^{-1}(\eta)|\underline{x})\frac{1}{\tau'(\tau^{-1}(\eta))}$, $(\tau'(\tau^{-1}(\eta)) \neq 0)$, and

$\frac{d}{d\eta}\tau'(\tau^{-1}(\eta)) = \frac{d}{d\tau^{-1}(\eta)}\tau'(\tau^{-1}(\eta))\frac{d\tau^{-1}(\eta)}{d\eta} = \tau''(\tau^{-1}(\eta))\frac{1}{\tau'(\tau^{-1}(\eta))}$, $(\tau'(\tau^{-1}(\eta)) \neq 0)$

so (**) : $L^{*''}(\eta|\underline{x}) = \frac{1}{[\tau'(\tau^{-1}(\eta))]^2}\left(L''(\tau^{-1}(\eta)|\underline{x}) - \frac{\tau''(\tau^{-1}(\eta))}{\tau'(\tau^{-1}(\eta))}L'(\tau^{-1}(\eta)|\underline{x})\right)$ #

3) $\because$ $\hat{\theta}$ is mle of θ . and $\eta = \tau(\theta)$ so by mle invariance,
$\hat{\eta} = \tau(\hat{\theta})$ is the mle of η #

4) $\because$ $\hat{\theta}$ is mle of θ, so $\begin{cases} 1. \frac{d}{d\theta}L(\theta|\underline{x})\big|_{\theta=\hat{\theta}} = 0 \\ 2. \frac{d^2}{d\theta^2}L(\theta|\underline{x})\big|_{\theta=\hat{\theta}} < 0 \end{cases}$

①
$\because L^{*'}(\eta|\underline{x})\big|_{\eta=\hat{\eta}} = L'(\tau^{-1}(\tau(\hat{\theta})))\frac{1}{\tau'(\tau^{-1}(\tau(\hat{\theta})))} = L'(\hat{\theta})\frac{1}{\tau'(\hat{\theta})} = 0$ $(\tau'(\hat{\theta}) \neq 0)$

② $\ell^{*''}(\eta|x)\Big|_{\eta=\hat{\eta}} = \dfrac{1}{[\tau'(\tau^{-1}(\tau(\hat\theta)))]^2}\left(\ell''(\tau^{-1}(\tau(\hat\theta))) - \dfrac{\tau''(\tau^{-1}(\tau(\hat\theta)))}{\tau'(\tau^{-1}(\tau(\hat\theta)))}\ell'(\tau^{-1}(\tau(\hat\theta)))\right)$

$$= \frac{1}{[\tau'(\hat\theta)]^2}\left(\ell''(\hat\theta) - \frac{\tau''(\hat\theta)}{\tau'(\hat\theta)}\ell'(\hat\theta)\right) = \frac{\ell''(\hat\theta)}{[\tau'(\hat\theta)]^2} < 0$$

by ①,②

$\therefore\ \hat\eta = \tau(\hat\theta)$ is the mle of η ✓.

+6

3. [+8] Let $\mathbf{X} \sim f(\mathbf{x}|\theta)$ and $W(\mathbf{X})$ be an estimator.

1) [+2]Prove $E_\theta\left[\frac{d}{d\theta}\log f(\mathbf{X}|\theta)\right] = 0$ (under some condition)

2) [+2]Prove $I_n(\theta) = E_\theta\left[\frac{d}{d\theta}\log f(\mathbf{X}|\theta)\right]^2 = -E_\theta\left[\frac{d^2}{d\theta^2}\log f(\mathbf{X}|\theta)\right]$

(under some condition)

3) [+2]Derive the Cramér–Rao inequality (under some condition)

4) [+2] Derive the Cramér–Rao inequality under the case of $X_1, \ldots, X_n \overset{iid}{\sim} f(x|\theta)$ and an unbiased estimator $E_\theta W(\mathbf{X}) = \theta$.

Suppose X is continuous r.v.

+2 1) $E_\theta\left(\frac{d}{d\theta}\ln f(x|\theta)\right) = \int_\Omega \frac{d}{d\theta}\ln f(x|\theta) \cdot f(x|\theta)\, dx$ —— $\langle * \rangle$

$\because \frac{d}{d\theta}\ln f(x|\theta) = \frac{\frac{d}{d\theta}f(x|\theta)}{f(x|\theta)}$, $\therefore \langle * \rangle = E_\theta\left(\frac{d}{d\theta}\ln f(x|\theta)\right) = \int_\Omega \frac{\frac{d}{d\theta}f(x|\theta)}{f(x|\theta)} f(x|\theta)\, dx$

$= \frac{d}{d\theta}\int_\Omega f(x|\theta)\, dx = \frac{d}{d\theta} 1 = 0$ #

+2

2) $\because -E_\theta\left[\frac{d^2}{d\theta^2}\ln f(x|\theta)\right] = -\int_\Omega \frac{d^2}{d\theta^2}\ln f(x|\theta) \cdot f(x|\theta)\, dx$ —— $\langle ** \rangle$

$\because \frac{d}{d\theta}\ln f(x|\theta) = \frac{\frac{d}{d\theta}f(x|\theta)}{f(x|\theta)}$, $\frac{d^2}{d\theta^2}\ln f(x|\theta) = \frac{\frac{d^2}{d\theta^2}f(x|\theta) \cdot f(x|\theta) - \left(\frac{d}{d\theta}f(x|\theta)\right)^2}{(f(x|\theta))^2}$

$\therefore \langle ** \rangle = -\int_\Omega \frac{\frac{d^2}{d\theta^2}f(x|\theta) f(x|\theta) - \left(\frac{d}{d\theta}f(x|\theta)\right)^2}{f(x|\theta)}\, dx = -\int_\Omega \frac{d^2}{d\theta^2}f(x|\theta)\, dx + \int_\Omega \frac{\left(\frac{d}{d\theta}f(x|\theta)\right)^2}{f(x|\theta)}\, dx$

$= -\frac{d^2}{d\theta^2}\int_\Omega f(x|\theta)\, dx + \int_\Omega \frac{\left(\frac{d}{d\theta}f(x|\theta)\right)^2}{f(x|\theta)}\, dx = -\frac{d^2}{d\theta^2} 1 + \int_\Omega \frac{\left(\frac{d}{d\theta}f(x|\theta)\right)^2}{f(x|\theta)}\, dx = \int_\Omega \frac{\left(\frac{d}{d\theta}f(x|\theta)\right)^2}{f(x|\theta)}\, dx$

$= \int_\Omega \left(\frac{\frac{d}{d\theta}f(x|\theta)}{f(x|\theta)}\right)^2 f(x|\theta)\, dx = \int_\Omega \left(\frac{d}{d\theta}\ln f(x|\theta)\right)^2 f(x|\theta)\, dx = E_\theta\left(\frac{d}{d\theta}\ln f(x|\theta)\right)^2$ #

+0

3) That $I(\theta) = E_\theta\left(\frac{d}{d\theta}\ln f(x|\theta)\right)^2$, then $Var_\theta(W(X)) \geq \frac{\left(\frac{d}{d\theta}E(W(X))\right)^2}{I(\theta)}$ #

Derive

4) $Var_\theta(W(X)) \geq \frac{1}{nI_1(\theta)}$, where $I_1(\theta) = E_\theta\left(\frac{d}{d\theta}\ln f(x|\theta)\right)^2$ #

+2

+5

4. [+5] Let $X_1,\ldots,X_{10} \overset{iid}{\sim} Bernoulli(p)$, and consider a hypothesis test for

$$H_0 : p \in \Theta_0 = \{p; 0 \le p \le 0.8\} \text{ vs. } H_1 : p \in \Theta_0^c = \{p; 0.8 < p \le 1\}$$

A statistical decision on the action space $A = \{a_0, a_1\}$ is defined as

$$\delta(\mathbf{x}) = \begin{cases} a_1 & \text{if } \sum_{i=1}^{10} x_i \ge 10 \\ a_0 & \text{if } \sum_{i=1}^{10} x_i < 10 \end{cases}$$

The 0-1 loss function is defined as

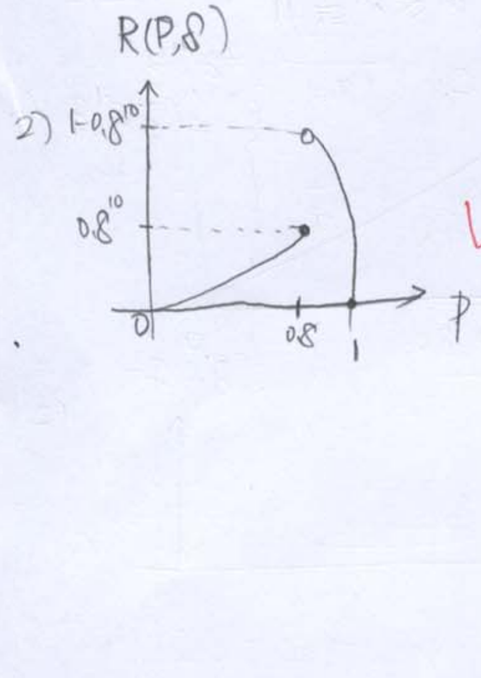
$$L(p,a_0) = \begin{cases} 0 & \text{if } p \in \Theta_0 \quad (0 \le p \le 0.8) \\ 1 & \text{if } p \in \Theta_0^c \quad (p > 0.8) \end{cases}, \quad L(p,a_1) = \begin{cases} 1 & \text{if } p \in \Theta_0 \quad (p \le 0.8) \\ 0 & \text{if } p \in \Theta_0^c \quad (p > 0.8) \end{cases}$$

1) [+3] Calculate the risk function $R(p,\delta)$.

2) [+2] Draw the graph of the risk function (include details).

1) $$R(p,\delta) = \begin{cases} P(\sum_{i=1}^{10} X_i < 10 \mid p > 0.8) = 1 - P(\sum_{i=1}^{10} X_i = 10 \mid p > 0.8) = 1 - p^{10} & ; p > 0.8 \\ P(\sum_{i=1}^{10} X_i \ge 10 \mid p \le 0.8) = P(\sum_{i=1}^{10} X_i = 10 \mid p \le 0.8) = p^{10} & ; p \le 0.8 \end{cases}$$

2)



+8 5. [+8] $X_1, \ldots, X_{10} \overset{iid}{\sim} N(\mu, \sigma^2)$, where $\theta = (\mu, \sigma^2)$ are unknown. Consider a test for $H_0 : \mu = \mu_0$ vs. $H_1 : \mu \neq \mu_0$.

1) [+2] Calculate the maximized likelihood $L(\hat{\mu}, \hat{\sigma}^2 \mid \mathbf{x})$ (need to be simplified)
2) [+2] Calculate the maximized likelihood $L(\mu_0, \hat{\sigma}_0^2 \mid \mathbf{x})$ under $H_0 : \mu = \mu_0$ (need to be simplified)
3) [+2] Calculate the LR statistics $\lambda(\mathbf{x})$
4) [+2] Derive the rejection region R (need to be simplified)

1) $f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$

$L(\underset{\sim}{x} \mid \mu, \sigma^2) = \prod_{i=1}^{10} f(x_i \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^5 \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{10}(x_i-\mu)^2\right)$

$\ln L(\underset{\sim}{x} \mid \mu, \sigma^2) = -5 \ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{10}(x_i-\mu)^2$,

$\frac{\partial}{\partial \mu} \ln L(\underset{\sim}{x} \mid \mu, \sigma^2) = -\frac{1}{2\sigma^2}\sum_{i=1}^{10}(x_i-\mu)(-2) = 0 \Rightarrow \hat{\mu} = \frac{1}{10}\sum_{i=1}^{10} x_i = \bar{x}$

$\frac{\partial}{\partial \sigma^2} \ln L(\underset{\sim}{x} \mid \mu, \sigma^2) = -5\frac{2\pi}{2\pi\sigma^2} + \frac{1}{\sigma^4 2}\sum_{i=1}^{10}(x_i-\mu)^2 = 0 \Rightarrow \sigma^2 = \frac{1}{10}\sum_{i=1}^{10}(x_i-\mu)^2 \Rightarrow \hat{\sigma}^2 = \frac{1}{10}\sum_{i=1}^{10}(x_i-\bar{x})^2$

and $\frac{\partial^2}{\partial \theta^2} \ln L(\underset{\sim}{x} \mid \mu, \sigma^2)\Big|_{\theta=\hat{\theta}} < 0$, where $\theta = (\mu, \sigma^2)$, $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$

$\therefore \begin{cases} \hat{\mu} = \bar{x} \\ \hat{\sigma}^2 = \frac{1}{10}\sum_{i=1}^{10}(x_i-\bar{x})^2 \end{cases}$ is mle of $\begin{matrix}\mu \\ \sigma^2\end{matrix}$ $\therefore L(\underset{\sim}{x} \mid \hat{\mu}, \hat{\sigma}^2) = \left(\frac{1}{2\pi\hat{\sigma}^2}\right)^5 \exp(-5)$

2) Under $\mu = \mu_0$, $\hat{\mu} = \mu_0$ is the mle of μ

$L(\underset{\sim}{x} \mid \sigma^2) = \prod_{i=1}^{10} f(x_i \mid \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^5 \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{10}(x_i-\mu_0)^2\right)$, $\ln L(\underset{\sim}{x} \mid \sigma^2) = -5 \ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{10}(x_i-\mu_0)^2$

$\frac{d}{d\sigma^2} \ln L(\underset{\sim}{x} \mid \sigma^2) = -5\frac{2\pi}{2\pi\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{10}(x_i-\mu_0)^2 = 0 \Rightarrow \hat{\sigma}_0^2 = \frac{1}{10}\sum_{i=1}^{10}(x_i-\mu_0)^2$

and $\frac{d^2}{d\sigma^4} \ln L(\underset{\sim}{x} \mid \sigma^2)\Big|_{\sigma^2=\hat{\sigma}_0^2} < 0$. $\therefore \hat{\sigma}_0^2$ is mle of σ^2.

$\therefore L(\underset{\sim}{x} \mid \mu_0, \hat{\sigma}_0^2) = \left(\frac{1}{2\pi\hat{\sigma}_0^2}\right)^5 \exp(-5)$

3) $\lambda(\underset{\sim}{x}) = \frac{\sup_{\theta \in \Theta_0} L(\underset{\sim}{x} \mid \mu, \sigma^2)}{\sup_{\theta \in \Theta} L(\underset{\sim}{x} \mid \mu, \sigma^2)} = \frac{\left(\frac{1}{2\pi\hat{\sigma}_0^2}\right)^5 \exp(-5)}{\left(\frac{1}{2\pi\hat{\sigma}^2}\right)^5 \exp(-5)} = \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^5 = \left(\frac{\sum_{i=1}^{10}(x_i-\bar{x})^2}{\sum_{i=1}^{10}(x_i-\mu_0)^2}\right)^5$

4). $\lambda(\underset{\sim}{x}) = \left(\dfrac{\sum_{i=1}^{10} (x_i - \bar{x})^2}{\sum_{i=1}^{10} (x_i - \mu_0)^2} \right)^5 = \left(\dfrac{\sum_{i=1}^{10} (x_i - \bar{x})^2}{\sum_{i=1}^{10} (x_i - \bar{x} + \bar{x} - \mu_0)^2} \right)^5 = \left(\dfrac{\sum_{i=1}^{10} (x_i - \bar{x})^2}{\sum_{i=1}^{10} (x_i - \bar{x})^2 + 10(\bar{x} - \mu_0)^2} \right)^5$

$= \left(\dfrac{1}{1 + \frac{10(\bar{x} - \mu_0)^2}{\sum_{i=1}^{10} (x_i - \bar{x})^2}} \right)^5 = \left(\dfrac{1}{1 + \frac{1}{9} \left(\frac{\bar{x} - \mu_0}{\frac{S}{\sqrt{10}}} \right)^2} \right)^5 < C$, where $S^2 = \frac{1}{9} \sum_{i=1}^{10} (x_i - \bar{x})^2$

let $T = \dfrac{\bar{x} - \mu_0}{\frac{S}{\sqrt{10}}} \overset{H_0}{\sim} t(9)$

$c^{\frac{1}{5}}$?

$\therefore \lambda(\underset{\sim}{x}) = \left(\dfrac{1}{1 + \frac{1}{9} T^2} \right)^5 < C \Leftrightarrow \dfrac{1}{1 + \frac{1}{9} T^2} < \sqrt[5]{C} \Leftrightarrow 1 < \sqrt[5]{C} \left(1 + \frac{1}{9} T^2 \right)$

$\Leftrightarrow \dfrac{1}{\sqrt[5]{C}} < 1 + \frac{1}{9} T^2 \Leftrightarrow \dfrac{1}{\sqrt[5]{C}} - 1 < \frac{1}{9} T^2 \Leftrightarrow T^2 > \dfrac{9}{\sqrt[5]{C}} - 9$

$\Leftrightarrow |T| > \sqrt{\dfrac{9}{\sqrt[5]{C}} - 9}$

$\therefore$ reject H_0 if $|T| > \sqrt{\dfrac{9}{\sqrt[5]{C}} - 9}$, C = constand #

, $T \overset{H_0}{\sim} t(9)$