Homework 4 3 $\frac{2}{2}$

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1. Let $X_1,\ldots,X_{10} \overset{iid}{\sim} Bernoulli(p)$, and consider a hypothesis test for

$$H_0 : p \in \Theta_0 = \{p;\, p > 0.5\} \quad \text{vs.} \quad H_1 : p \in \Theta_0^c = \{p;\, p \le 0.5\}$$

A statistical decision on the action space $A = \{a_0, a_1\}$ is defined as

$$\delta(\mathbf{x}) = \begin{cases} a_1 & \text{if } \sum_{i=1}^{10} x_i < 3 \\ a_0 & \text{if } \sum_{i=1}^{10} x_i \ge 3 \end{cases}$$

The 0-1 loss function is defined as

$$L(p,a_0) = \begin{cases} 0 & \text{if } p \in \Theta_0 \\ 1 & \text{if } p \in \Theta_0^c \end{cases}, \qquad L(p,a_1) = \begin{cases} 1 & \text{if } p \in \Theta_0 \\ 0 & \text{if } p \in \Theta_0^c \end{cases}$$

1) Calculate the risk function $R(p,\delta)$

2) Draw the graph of the risk function (include details).

Check your graph by computer.

<sol>

1) $p \in \Theta_0$ $(p > 0.5)$

$$R(p,\delta) = EL(p,\delta(\mathbf{X})) = P(\delta(\mathbf{X}) = a_1) = P(\sum_{i=1}^{10} x_i < 3) \text{ where } \sum_{i=1}^{10} x_i \sim Binomial(10,p)$$

$$= C_0^{10} p^0 (1-p)^{10} + C_1^{10} p^1 (1-p)^9 + C_2^{10} p^2 (1-p)^8$$

$$= (1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^8$$

$$= (1-p)^8((1-p)^2 + 10p(1-p) + 45p^2))$$

$$= (1-p)^8(1-2p+p^2+10p-10p^2+45p^2)$$

$$= (1-p)^8(36p^2+8p+1)$$

$p \in \Theta_0^c$ $(p \le 0.5)$

$$R(p,\delta) = P(\delta(\mathbf{X}) = a_0) = P(\sum_{i=1}^{10} x_i \ge 3) = 1 - P(\sum_{i=1}^{10} x_i < 3) = 1-(1-p)^8(36p^2+8p+1)$$

$$\therefore R(p,\delta) = \begin{cases} (1-p)^8(36p^2+8p+1) & \text{if } p \in \Theta_0 = \{p;\, p > 0.5\} \\ 1-(1-p)^8(36p^2+8p+1) & \text{if } p \in \Theta_0^c = \{p;\, p \le 0.5\} \end{cases}$$

2) Under $p \in \Theta_0$ $(p > 0.5)$.

$R(p, \delta) = (1-p)^8 (36p^2 + 8p + 1)$

$$\frac{\partial}{\partial p}\left((1-p)^8(36p^2+8p+1)\right) = -8(1-p)^7(36p^2+8p+1) + (1-p)^8(72p+8)$$
$$= (1-p)^7(-8(36p^2+8p+1) + (1-p)(72p+8))$$
$$= (1-p)^7(-288p^2-64p-8+72p+8-72p^2-8p)$$
$$= (1-p)^7(-360p^2) < 0 \quad \text{in} \quad 0.5 < p \le 1$$

↳ decreasing ↘ or ↘

$$\frac{\partial^2}{\partial p^2}\left((1-p)^8(36p^2+8p+1)\right) = -7(1-p)^6(-360p^2) + (1-p)^7(-720p)$$
$$= (1-p)^6(2520p^2 + (1-p)(-720p))$$
$$= (1-p)^6(3240p^2 - 720p).$$
$$= (1-p)^6 p(3240p - 720) > 0 \quad \text{in} \quad 0.5 < p \le 1$$

↳ concave upward ∪

Under $p \in \Theta_0^c$ $(p \le 0.5)$.

$R(p, \delta) = 1 - (1-p)^8(36p^2+8p+1)$

$$\frac{\partial}{\partial p}\left(1-(1-p)^8(36p^2+8p+1)\right) = (1-p)^7(360p^2) > 0 \quad \text{in} \quad 0 \le p \le 0.5$$

↳ increasing ↗ or ↗

$$\frac{\partial^2}{\partial p^2}\left(1-(1-p)^8(36p^2+8p+1)\right) = (1-p)^6 p(-3240p+720) \overset{\text{let}}{=} 0$$

$\Rightarrow p = 0, \underline{1}, \frac{2}{9}$

↳ unreasonable $\because 0 \le p \le 0.5$

(+1)

$$\frac{\partial^3}{\partial p^3}\left(1-(1-p)^8(36p^2+8p+1)\right) = -6(1-p)^5(-3240p^2+720p) + (1-p)^6(-6480p+720)$$
$$= (1-p)^5(19440p^2 - 4320p + (1-p)(-6480p+720))$$
$$= (1-p)^5(25920p^2 - 11520p + 720)$$

$\Rightarrow \frac{\partial^3}{\partial p^3}\left(1-(1-p)^8(36p^2+8p+1)\right)\Big|_{p=\frac{2}{9}} \fallingdotseq -159.3916916 \neq 0.$

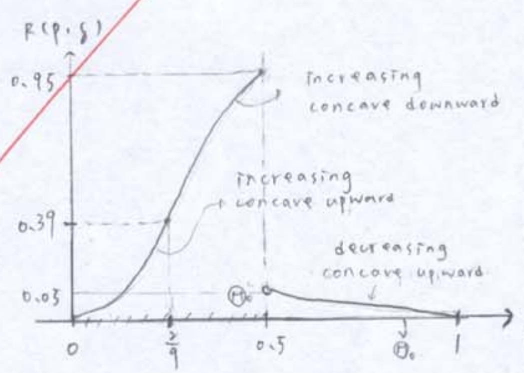
$\therefore p = \frac{2}{9}$ is a inflection point. $\Rightarrow R(\frac{2}{9}, \delta) \fallingdotseq 0.38992$

$\Rightarrow \frac{\partial^2}{\partial p^2}\left(1-(1-p)^8(36p^2+8p+1)\right)\Big|_{p=0.1} \fallingdotseq 21.0450636 > 0.$

$\therefore$ In $0 \le p \le \frac{2}{9}$, concave upward.

$\Rightarrow \frac{\partial^2}{\partial p^2}\left(1-(1-p)^8(36p^2+8p+1)\right)\Big|_{p=0.3} \fallingdotseq -8.8942644 < 0$

$\therefore$ In $p \le \frac{2}{9} \le 0.5$, concave downward.



2)

<table>
<tr><th>R-code</th><th>Output</th></tr>
<tr><td>
<pre>
f1 = function(p0){
((1-p0)^8)*(36*(p0^2)+8*p0+1)
}
p0 = seq(0.51, 1, by = 0.01)
plot(p0, f1(p0), type = "l", xlim = c(0,1), ylim =
c(0,1), xlab = "p", ylab = "Risk function")

p1 = c()
f2 = function(p1){
1-((1-p1)^8)*(36*(p1^2)+8*p1+1)
}
curve(f2, 0, 0.5, lty = 2, add = TRUE)

legend(0.7, 1, legend = c("Under H0", "Under
H1"), lty = c(1, 2))
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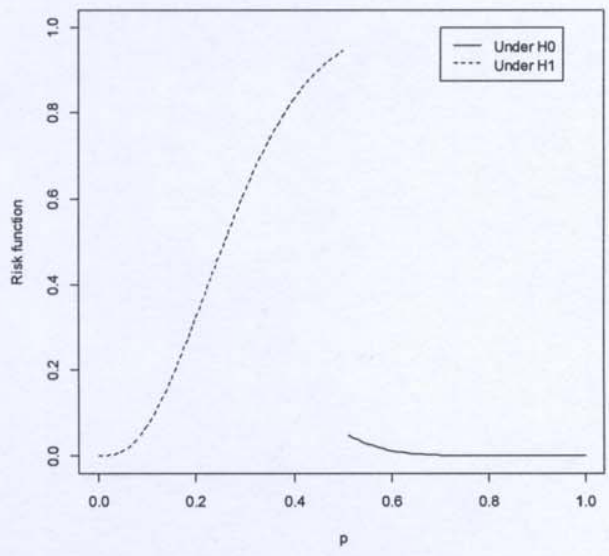

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