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7.19 Suppose that the random variables $Y_1, \dots, Y_n$ satisfy 10/10

$$Y_i = \beta X_i + \varepsilon_i \quad , \quad i = 1, \dots, n$$

where $x_1, \dots, x_n$ are fixed constants, and $\varepsilon_1, \dots, \varepsilon_n$ are iid $n(0, \sigma^2)$, σ^2 unknown.

(a) Find a two-dimensional sufficient statistic for (β, σ^2)

Ans: $\because Y_i = \beta X_i + \varepsilon_i \sim N(\beta X_i, \sigma^2)$

$$\therefore L(\theta | \underline{Y}) = \prod_{i=1}^{n} f(y_i | \theta) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-\frac{1}{2}} \cdot \exp\left(-\frac{(Y_i - \beta X_i)^2}{2\sigma^2}\right)$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \cdot \exp\left(-\sum_{i=1}^{n} \frac{(Y_i - \beta X_i)^2}{2\sigma^2}\right)$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \cdot \exp\left(-\frac{\beta^2 X_i^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} Y_i^2 - \beta \sum_{i=1}^{n} Y_i X_i\right)\right), \quad \theta = (\beta, \sigma^2)$$

Let $h(\underline{Y}) = 1$,

$$g\left(\sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} X_i Y_i ; \beta, \sigma^2\right) = (2\pi\sigma^2)^{-\frac{n}{2}} \cdot \exp\left(-\frac{\beta^2 X_i^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} Y_i^2 - \beta \sum_{i=1}^{n} X_i Y_i\right)\right)$$

$\Rightarrow$ By Neyman Fisher factorization therorem

$\left(\sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} X_i Y_i\right)$ is a sufficient statistic for (β, σ^2).

(b) Find the MLE of β, and show that it is an unbiased estimator of β.

Ans: By (a)

$$\log L(\theta | \underline{Y}) = -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (Y_i^2 - 2\beta X_i Y_i + \beta^2 X_i^2)$$

$$\Rightarrow \frac{\partial \log L(\theta | \underline{Y})}{\partial \beta} = \frac{1}{\sigma^2} \sum_{i=1}^{n} X_i Y_i - \frac{\beta}{\sigma^2} \sum_{i=1}^{n} X_i^2 = 0 \quad \Rightarrow \hat{\beta} = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2}$$

$$\frac{\partial^2 \log L(\theta | \underline{Y})}{\partial \beta^2} = -\frac{1}{\sigma^2} \sum_{i=1}^{n} X_i^2 < 0 \quad \Rightarrow \hat{\beta} \text{ is the MLE of } \beta.$$

$$E[\hat{\beta}] = E\left[\frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2}\right] = \frac{\sum_{i=1}^{n} X_i E[Y_i]}{\sum_{i=1}^{n} X_i^2} = \frac{\beta \sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n} X_i^2} = \beta$$

$\Rightarrow \hat{\beta}$ is an unbiased estimator of β.

(c) Find the distribution of the MLE of β.

Ans: $\because Y_i \sim N(\beta X_i, \sigma^2)$ and X_i is constant.

From (b) $E[\hat{\beta}] = \beta$

$$Var(\hat{\beta}) = Var\left(\frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2}\right) = \left(\frac{1}{\sum_{i=1}^{n} X_i^2}\right)^2 \sum_{i=1}^{n} Var(X_i Y_i) = \left(\frac{1}{\sum_{i=1}^{n} X_i^2}\right)^2 \sigma^2 \left(\sum_{i=1}^{n} X_i^2\right) = \frac{\sigma^2}{\sum_{i=1}^{n} X_i^2}$$

$$\therefore \hat{\beta} = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2} \sim N\left(\beta, \frac{\sigma^2}{\sum_{i=1}^{n} X_i^2}\right)$$

7.20 Consider $Y_1, \ldots, Y_n$ as defined in Exercise 7.19

(a) Show that $\dfrac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}$ is an unbiased estimator of β.

Ans: $$E\left[\frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}\right] = \frac{1}{\sum_{i=1}^n X_i} E\left[\sum_{i=1}^n Y_i\right] = \frac{1}{\sum_{i=1}^n X_i} \cdot \beta \sum_{i=1}^n X_i = \beta$$

$$\Rightarrow \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i} \text{ is an unbiased estimator of } \beta.$$

(b) Calculate the exact variance of $\dfrac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}$ and compare it to the variance of the MLE.

Ans:
$$Var\left(\frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}\right) = \left(\frac{1}{\sum_{i=1}^n X_i}\right)^2 \cdot \sum_{i=1}^n Var(Y_i) = \left(\frac{1}{\sum_{i=1}^n X_i}\right)^2 \cdot n\sigma^2 = \frac{n\sigma^2}{(n\bar{x})^2} = \frac{\sigma^2}{n\bar{x}^2}$$

$$\because \sum_{i=1}^n X_i^2 - n\bar{x}^2 = \sum_{i=1}^n (X_i - \bar{x})^2 \geq 0 \Rightarrow \sum_{i=1}^n X_i^2 \geq n\bar{x}^2$$

$$\therefore Var(\hat{\beta}) = \frac{\sigma^2}{\sum_{i=1}^n X_i^2} \leq \frac{\sigma^2}{n\bar{x}^2} = Var\left(\frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}\right)$$

7.21 Again, let $Y_1, \ldots, Y_n$ be as defined in Exercise 7.19

(a) Show that $\left[\dfrac{\sum_{i=1}^n \left(\frac{Y_i}{X_i}\right)}{n}\right]$ is also an unbiased estimator of β.

Ans: $$E\left[\frac{\sum_{i=1}^n \left(\frac{Y_i}{X_i}\right)}{n}\right] = \frac{1}{n}\sum_{i=1}^n \frac{E[Y_i]}{X_i} = \frac{1}{n}\sum_{i=1}^n \frac{\beta X_i}{X_i} = \frac{1}{n}(n\beta) = \beta$$

$$\Rightarrow \frac{\sum_{i=1}^n \left(\frac{Y_i}{X_i}\right)}{n} \text{ is an unbiased estimator of } \beta.$$

(b) Calculate the exact variance of $\dfrac{\sum_{i=1}^n \left(\frac{Y_i}{X_i}\right)}{n}$ and compare it to the variance of the estimators in the previous two exercises.

Ans: $$Var\left[\frac{\sum_{i=1}^n \left(\frac{Y_i}{X_i}\right)}{n}\right] = \frac{1}{n^2}\sum_{i=1}^n \frac{1}{X_i^2} Var(Y_i) = \sum_{i=1}^n \frac{\sigma^2}{n^2 X_i^2}$$

We want to find the relation of $\sum_{i=1}^n \dfrac{\sigma^2}{n^2 X_i^2}$, $\dfrac{\sigma^2}{n\bar{x}^2}$ and $\dfrac{\sigma^2}{\sum_{i=1}^n X_i^2}$.

① By Cauchy inequlity

$$\left[\left(\frac{\sigma}{nX_1}\right)^2 + \left(\frac{\sigma}{nX_2}\right)^2 + \ldots + \left(\frac{\sigma}{nX_n}\right)^2\right]\left[(1)^2 + (1)^2 + \ldots + (1)^2\right] \geq \left[\left(\frac{\sigma}{nX_1}\right)(1) + \ldots + \left(\frac{\sigma}{nX_n}\right)(1)\right]^2$$

$$\Rightarrow \left(\sum_{i=1}^{n} \frac{\sigma^2}{n^2 X_i^2}\right) n \geq \left[\sum_{i=1}^{n} \frac{\sigma}{n X_i}\right]^2 \Rightarrow \sum_{i=1}^{n} \frac{\sigma^2}{n^2 X_i^2} \geq \frac{\left[\sum_{i=1}^{n} \frac{\sigma}{n X_i}\right]^2}{n}$$

② We want to proof $\dfrac{\left[\sum_{i=1}^{n} \frac{\sigma}{n X_i}\right]^2}{n} \geq \dfrac{n\sigma^2}{\left(\sum_{i=1}^{n} X_i\right)^2} = \dfrac{\sigma^2}{n\bar{X}^2}$

$\because$ By harmonic inequlity

$$\frac{\sum_{i=1}^{n} X_i}{n} \geq \frac{n}{\sum_{i=1}^{n} \frac{1}{X_i}} \Rightarrow \left(\frac{\sum_{i=1}^{n} X_i}{n}\right)^2 \geq \frac{n^2}{\left(\sum_{i=1}^{n} \frac{1}{X_i}\right)^2}$$

$$\Rightarrow \bar{X}^2 \geq \frac{n^2}{\left(\sum_{i=1}^{n} \frac{1}{X_i}\right)^2} \Rightarrow \left[\frac{\sum_{i=1}^{n} \frac{1}{X_i}}{n}\right]^2 \geq \frac{1}{\bar{X}^2}$$

$$\therefore \frac{\left[\sum_{i=1}^{n} \frac{\sigma}{n X_i}\right]^2}{n} = \frac{\left[\sum_{i=1}^{n} \frac{1}{X_i}\right]^2}{n} \cdot \left(\frac{\sigma^2}{n^2}\right) \geq \frac{1}{\bar{X}^2} \cdot \left(\frac{\sigma^2}{n}\right) = \frac{\sigma^2}{n\bar{X}^2}$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\sigma^2}{n^2 X_i^2} \geq \frac{\left[\sum_{i=1}^{n} \frac{\sigma}{n X_i}\right]^2}{n} \geq \frac{\sigma^2}{n\bar{X}^2}$$

③ from (b) and ②

We know $\displaystyle\sum_{i=1}^{n} \frac{\sigma^2}{n^2 X_i^2} \geq \frac{\sigma^2}{n\bar{X}^2} \geq \frac{\sigma^2}{\sum_{i=1}^{n} X_i^2}$,

that is $\displaystyle Var\left(\frac{\sum_{i=1}^{n}\left(\frac{Y_i}{X_i}\right)}{n}\right) \geq Var\left(\frac{\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} X_i}\right) \geq Var\left(\frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2}\right)$

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