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Ex 7.6: (a).

Let $X_1, \cdots, X_n \overset{iid}{\sim} f(x_i|\theta) = \theta x_i^{-2}$; $0 < \theta \le x_i < \infty$, $i = 1, 2, \cdots, n$

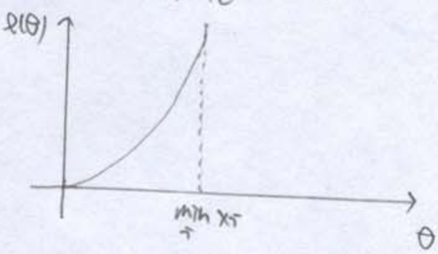
$$f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \theta \cdot x_i^{-2} \cdot I_{[\theta,\infty)}(x_i) = \underbrace{\theta^n \cdot I_{\{\min x_i \ge \theta,\ \forall i\}}}_{g(T(\underset{\sim}{x})|\theta)} \cdot \underbrace{\prod_{i=1}^{n} x_i^{-2}}_{h(\underset{\sim}{x})}$$

$T(\underset{\sim}{X}) = \min_i X_i$, By the Factorization thm, $T(\underset{\sim}{X}) = \min_i X_i$ is a sufficient statistic for θ

(b).

$$\ell(\theta) = \prod_{i=1}^{n} \theta \cdot x_i^{-2} \cdot I_{[\theta,\infty)}(x_i) = \theta^n \ I_{\{\min x_i \ge \theta,\ \forall i\}} \prod_{i=1}^{n} x_i^{-2}$$

$\ell(\theta)$ is a increasing function in $\{\min_i x_i \ge \theta > 0\}$



Therefore the MLE of θ is $\hat{\theta} = \min_i X_i$

(c).

$$E(X) = \int_{\theta}^{\infty} x \cdot \theta \cdot x^{-2}\, dx = \theta \cdot \int_{\theta}^{\infty} x^{-1}\, dx = \theta \cdot [\ln x]_{\theta}^{\infty}$$

$\therefore E(X)$ does not exist. and M.M.E does not exist either.

Ex. 7.8: $X \sim N(0, \sigma^2)$, $\sigma^2 > 0$, $-\infty < x < \infty$

(a).

$$Var(X) = E(X^2) - (E(X))^2 = E(X^2) - 0 = E(X^2) = \sigma^2$$

Therefore X^2 is an unbiased estimator of σ^2

(b).

$$\ell(\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$$\log \ell(\sigma) = \frac{-1}{2}\log 2\pi - \log \sigma - \frac{x^2}{2\sigma^2}$$

$$\frac{d \log \ell(\sigma)}{d\sigma} = \frac{-1}{\sigma} + \frac{x^2}{\sigma^3} \overset{let}{=} 0$$

$\Rightarrow \hat{\sigma}^2 = x^2 \quad \Rightarrow \hat{\sigma} = |X| \quad \because \sigma > 0$

$$\frac{d^2 \log \ell(\sigma)}{d\sigma^2} = \frac{1}{\sigma^2} - \frac{3x^2}{\sigma^4}\Big|_{\sigma=|X|} = \frac{1}{x^2} - \frac{3x^2}{x^4} = \frac{-2}{x^2} < 0 \quad \text{for all } x$$

Therefore the MLE of σ is $\hat{\sigma} = |X|$

(c).

$E(X) = 0$

$E(X^2) = 0 + \sigma^2$

$\Rightarrow \hat{\sigma}^2 = X^2 \quad \Rightarrow \hat{\sigma} = |X| \quad (\because \sigma > 0)$

Therefore the M.M.E of σ is $\hat{\sigma} = |X|$