Quiz#2, Mathematical Statistics I, 2013 Fall

Name: 陳愛群 AI-CHUN CHEN 102225014.

+2/2.

9/10.

1. Let X be a random variable. Prove that

1) If $E[X]=0$ and $X \ge 0$, then $X=0$.

2) $Cov(X,X)=Var(X)$.

1) $E[X] = \sum_i x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + \cdots + x_n f(x_n) = 0$.

↓ consider discrete type

and $f(x_1)+f(x_2)+\cdots+f(x_n)=1$ with $\underline{X \ge 0}$. ?

$x_i \ge 0$ for $i=1 \dots n$.

$\Rightarrow X = 0$ #

2) $Cov(X,X) = E(X-\mu_X)(X-\mu_X) = E(X-\mu_X)^2 = Var(X)$ #

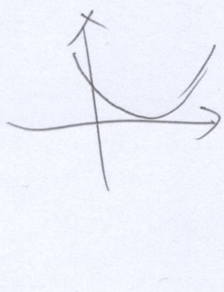
Let (X,Y) be a bivariate random vector. Prove

3) If $X \perp Y$, then $\rho_{XY}=0$.

4) $-1 \le \rho_{XY} \le 1$.

3) $\rho_{XY} = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y} = \dfrac{EXY - EX\,EY}{\sigma_X \sigma_Y} \underset{\text{by } X \perp Y}{=} \dfrac{EX\,EY - EX\,EY}{\sigma_X \sigma_Y} = 0$ #

4) Let $h(t) = E\left[(X-\mu_X)t + (Y-\mu_Y)\right]^2$



$= E\left[(X-\mu_X)^2 t^2 + 2t(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2\right]$

$= t^2 \sigma_X^2 + 2t\,Cov(X,Y) + \sigma_Y^2 \ge 0$ by the definition of expectation of square.

$\Rightarrow b^2 - 4ac = (2Cov(X,Y))^2 - 4\sigma_X^2 \sigma_Y^2 \le 0$.

$\Rightarrow Cov(X,Y)^2 \le \sigma_X^2 \sigma_Y^2$

$\Rightarrow -\sigma_X \sigma_Y \le Cov(X,Y) \le \sigma_X \sigma_Y$

$\Rightarrow -1 \le \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y} \le 1$

$\Rightarrow -1 \le \rho_{XY} \le 1$ #

$$\frac{Cov(X,Y)}{\sigma_X \sigma_Y} = -1 \qquad t = -1 \qquad \sigma_X^2 t + \sigma_X^2 = 0$$

2. A bivariate random vector (X,Y) has $EX = \mu_X$, $EY = \mu_Y$, $Var(X) = \sigma_X^2$, $Var(Y) = \sigma_Y^2$ and the correlation $\rho_{XY} = -1$.

+2/2

$-\sigma_X^2$ $\qquad \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = -1$

1) Derive the linear relationship between X and Y.

2) Simplify the above formula for the case of $\mu_X = \mu_Y$ and $\sigma_X^2 = \sigma_Y^2$.

$\frac{Cov(X,Y)}{\sigma_Y^2} = -1$

1) $\frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \rho_{XY} = -1 \Rightarrow Cov(X,Y) = -\sigma_X \sigma_Y \leq 0.$

Consider $h(t) = E\left[(X-\mu_X)t + (Y-\mu_Y)\right]^2 = t^2\sigma_X^2 + 2t\,Cov(X,Y) + \sigma_Y^2 \geq 0.$

$\Rightarrow \exists t$ s.t. $h(t) = 0.$

$\Rightarrow h'(t) = 2\sigma_X^2 t + 2\,Cov(X,Y) = 0.$

$\therefore$ When $t = \frac{-Cov(X,Y)}{\sigma_X^2}$, $h(t) = 0.$

$h(t) = 0 \Rightarrow (X-\mu_X)t + (Y-\mu_Y) = 0$

$\Rightarrow Y = -tX + \mu_X t + \mu_Y$

where $t = \frac{-Cov(X,Y)}{\sigma_X^2}$

$\Rightarrow Y = aX + b.$

where $\begin{cases} a = -t = \frac{Cov(X,Y)}{\sigma_X^2} \leq 0 \\ b = \mu_X t + \mu_Y \end{cases}$

$= \mu_X \frac{Cov(X,Y)}{\sigma_X^2} + \mu_Y.$

2) (i) $Cov(X,Y) = -\sigma_X^2 = -\sigma_Y^2.$

$\Rightarrow a = -1$

(ii) $b = \mu_X t + \mu_X = \mu_X\left(1 + \frac{-Cov(X,Y)}{\sigma_X^2}\right) = \mu_X(1+1) = 2\mu_X$

$\Rightarrow Y = -X + 2\mu_X$

#

$E(X) = \mu_X$.

$Var(U) = E(U - \mu_U)^2$

$\frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho}} \exp\left(-\frac{1}{2-\rho}(X\ Y)\Sigma^{-1}\begin{pmatrix}X\\Y\end{pmatrix}\right)$

$\Sigma = \begin{bmatrix}\sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2\end{bmatrix}$.

+2/2

3. Let (μ_X, μ_Y, σ_X^2, σ_Y^2, ρ) be the parameters of the bivariate normal random variable (X,Y). Let $\begin{bmatrix}U\\V\end{bmatrix} = \begin{bmatrix}a & b\\ b & c\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}$, where a, b, c are constant.

1) Derive $Var(U)$ and $Var(V)$.

2) Derive $Cov(U,V)$.

1) $\begin{bmatrix}U\\V\end{bmatrix} = \begin{bmatrix}aX+bY\\bX+cY\end{bmatrix} \Rightarrow \begin{cases} Var(U) = Var(aX+bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X,Y) \\ Var(V) = Var(bX+cY) = b^2Var(X) + c^2Var(Y) + 2bcCov(X,Y)\end{cases}$

$\Rightarrow \begin{cases} Var(U) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y \\ Var(V) = b^2\sigma_X^2 + c^2\sigma_Y^2 + 2bc\rho\sigma_X\sigma_Y \end{cases}$ ※

2) $Cov(U,V) = Cov(aX+bY, bX+cY) = ab\,Var(X) + ac\,Cov(X,Y) + b^2Cov(X,Y) + bc\,Var(Y)$

$= ab\sigma_X^2 + (ac+b^2)\rho\sigma_X\sigma_Y + bc\sigma_Y^2$ ※

+2/2

$f(x_i) = \frac{1}{\beta}e^{-\frac{x_i}{\beta}}$

4. 1) Calculate $P(X_1 > 2, \ldots, X_n > 2)$ when $X_1, \ldots, X_n \overset{iid}{\sim} \text{exponential}(\beta)$, where $\beta = EX_1$.

$0, \infty$.

$\lambda e^{-\lambda x}$. $\frac{1}{\lambda}$

2) Calculate $P(0 \le X_1 \le 2, \ldots, 0 \le X_n \le 2)$ when $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Use

CDF

$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}u^2)du$ to write the answer.

1) $P(X_1>2, \ldots, X_n>2) = P(X_1>2)P(X_2>2)\cdots P(X_n>2) = [1-F_{X_1}(2)][1-F_{X_2}(2)]\cdots[1-F_{X_n}(2)]$

$\overset{iid}{=} [1-F_{X_1}(2)]^n$ where $F_{X_1}(x) = 1 - e^{-\frac{x}{\beta}}$; $0<x<\infty$

$\Rightarrow [1-F_{X_1}(2)]^n = e^{-\frac{2n}{\beta}}$ ※

2) $P(0 \le X_1 \le 2) = P(X_1 \le 2) - P(X_1 \le 0) = P(\frac{X_1-\mu}{\sigma} \le \frac{2-\mu}{\sigma}) - P(\frac{X_1-\mu}{\sigma} \le \frac{-\mu}{\sigma})$

$= \Phi(\frac{2-\mu}{\sigma}) - \Phi(\frac{-\mu}{\sigma})$.

$\Rightarrow P(0 \le X_1 \le 2, \ldots, 0 \le X_n \le 2) = [\Phi(\frac{2-\mu}{\sigma}) - \Phi(\frac{-\mu}{\sigma})]^n$ ※

$+1/2$

5. A bivariate random vector (X,Y) has a bivariate normal distribution with parameters $(\mu_X,\ \mu_Y,\ \sigma_X^2,\ \sigma_Y^2,\ \rho)$. Calculate EX^2Y.

$$E X^2 Y = E \frac{\partial^3}{\partial s^2 \partial t}\left[e^{sX+tY}\right] = \frac{\partial^3}{\partial s^2 \partial t} E\left(e^{sX+tY}\right)$$

$$\checkmark = \frac{\partial^3}{\partial s^2 \partial t} M_{sX+tY}(1)\Big|_{s=t=0} = \frac{\partial^3}{\partial s^2 \partial t} \exp\left(s\mu_X + t\mu_Y + \frac{1}{2}(s^2\sigma_X^2 + 2st\rho\sigma_X\sigma_Y + t^2\sigma_Y^2)\right)\Big|_{s=t=0}$$

$+1$

$$= \frac{\partial^2}{\partial s^2} (\mu_Y + s\rho\sigma_X\sigma_Y + t\sigma_Y^2) M_{sX+tY}(1)\Big|_{s=t=0}$$

$$= \frac{\partial}{\partial s}\left(\rho\sigma_X\sigma_Y M_{sX+tY}(1) + (\mu_Y + s\rho\sigma_X\sigma_Y + t\sigma_Y^2)(\mu_X + s\sigma_X^2 + t\rho\sigma_X\sigma_Y)\right)\Big|_{s=t=0}$$

$$= \rho\sigma_X\sigma_Y (\mu_X + s\sigma_X^2 + t\rho\sigma_X\sigma_Y) + (\rho\sigma_X\sigma_Y)(\mu_X + s\sigma_X^2 + t\rho\sigma_X\sigma_Y) + (\mu_Y + s\rho\sigma_X\sigma_Y + t\sigma_Y^2)\sigma_X^2 \Big|_{s=t=0}$$

$$= \rho\sigma_X\sigma_Y\mu_X + \rho\sigma_X\sigma_Y\mu_X + \mu_Y\sigma_X^2$$

$$= 2\rho\sigma_X\sigma_Y\mu_X + \mu_Y\sigma_X^2 \ \#$$ ✗

should be.

$$\mu_X(\sigma_X^2 + \mu_X^2) + 2\sigma_X\sigma_Y\rho\mu_X.$$