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Quiz#1, Mathematical Statistics I, 2013 Fall

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1. A bivariate random vector (X,Y) has the pdf

$$f(x,y)=\begin{cases} cxy^2 & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases},$$

where c is a constant.

1) Find c such that $f(x,y)$ becomes a pdf.

2) Calculate $P(X+Y>1)$.

1) $\int_0^1 \int_0^1 cxy^2 \, dx\, dy = \int_0^1 c \left[\frac{x^2}{2}\right]_0^1 y^2 \, dy$

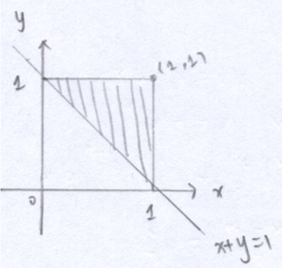
$= \int_0^1 \frac{c}{2} y^2 \, dy$

$= \frac{c}{2}\left[\frac{y^3}{3}\right]_0^1 = \frac{c}{6} = 1$

$\Rightarrow c = 6$

Hence, c is 6 s.t. $f(x,y)$ becomes a pdf. □

2)



$P(X+Y>1) = \int_0^1 \int_{1-y}^1 6xy^2 \, dx\, dy$

$= \int_0^1 6y^2 \left[\frac{x^2}{2}\right]_{1-y}^1 dy$

$\left(1^2-(1-y)^2 = 1-(1-2y+y^2) = 2y-y^2\right)$

$= \int_0^1 \left[6y^3 - 3y^4\right] dy$

$= \left[\frac{3}{2}y^4 - \frac{3}{5}y^5\right]_0^1$

$= \frac{3}{2} - \frac{3}{5} = \frac{15-6}{10} = \frac{9}{10}$

Hence, $P(X+Y>1) = \frac{9}{10}$. □

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2. Let $f(x,y)$ be the pdf of a random vector (X,Y). Suppose $X \perp Y$.

1) Prove that $Eg(X)h(Y) = Eg(X)Eh(Y)$.

2) Let $f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$ and $f_Y(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$,

Then, compute EX^2Y.

1) By pmf

$$E\, g(X)h(Y) = \sum_{x\in R}\sum_{y\in R} g(x)h(y)f(x,y)$$

$$= \sum_{x\in R}\sum_{y\in R} g(x)h(y)f_X(x)f_Y(y) \quad (\because X \perp Y)$$

$$= \sum_{x\in R} g(x)f_X(x)\sum_{y\in R} h(y)f_Y(y)$$

$$= E\,g(X)\cdot E\,h(Y)$$

By pdf

$$E\,g(X)h(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)\,dx\,dy \quad (\because X \perp Y)$$

$$= \int_{-\infty}^{\infty} g(x)f_X(x)\,dx \int_{-\infty}^{\infty} h(y)f_Y(y)\,dy$$

$$= E\,g(X)\,E\,h(Y)$$

Hence, $E\,g(X)h(Y) = Eg(X)\,E\,h(Y)$

2) we check $f_X(x)$ and $f_Y(y)$ are pdf.

$$\left.\begin{aligned} \int_0^{\infty} e^{-x}dx &= [-e^{-x}]_0^{\infty} = 1 \\ \int_0^{\infty} e^{-y}dy &= [-e^{-y}]_0^{\infty} = 1 \end{aligned}\right\} \Rightarrow f_X(x) \text{ and } f_Y(y) \text{ are pdf.}$$

$$EX^2 = \int_0^{\infty} x^2 e^{-x}\,dx \qquad \begin{pmatrix} u = x^2 & dv = e^{-x}dx \\ du = 2x\,dx & v = -e^{-x} \end{pmatrix} \qquad \begin{pmatrix} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x} \end{pmatrix}$$

$$= -x^2e^{-x}\Big|_0^{\infty} + 2\int_0^{\infty} xe^{-x}\,dx, \quad 其中 \int_0^{\infty} xe^{-x}\,dx = -xe^{-x}\Big|_0^{\infty} + \int_0^{\infty} e^{-x}\,dx$$

$$= [-e^{-x}]_0^{\infty} = 1$$

$$= 0 + 2\times 1 = 2$$

$$EY = \int_0^{\infty} y e^{-y}\,dy = -ye^{-y}\Big|_0^{\infty} + \int_0^{\infty} e^{-y}\,dy$$

$$= 0 + [-e^{-y}]_0^{\infty} = 0 + 1 = 1$$

By 1) $EX^2Y = EX^2\,EY = 2\times 1 = 2$

Hence, $EX^2Y = 2$ ▫

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3. Let (X,Y) be a discrete random vector with the marginal pmf

$$f_X(x)=\begin{cases} x/6, & x=1,2,3 \\ 0, & \text{otherwise} \end{cases}, \quad \text{and} \quad f_Y(y)=\begin{cases} y/6, & y=1,2,3 \\ 0, & \text{otherwise} \end{cases}.$$

1) Find a pmf $f(x,y)$ that is independent ($X \perp Y$).
2) Find a pmf $f(x,y)$ that is not independent.

(need to prove that $f(x,y)$ is a pmf)

1) $\because X \perp Y$

$\therefore f(x,y) = f_X(x) f_Y(y) = \frac{x}{6}\cdot\frac{y}{6} = \frac{xy}{36}$, $x=1,2,3$, $y=1,2,3$

$$\sum_{x\in\{1,2,3\}} \sum_{y\in\{1,2,3\}} \frac{xy}{36} = \frac{1\times1 + 1\times2 + 1\times3 + 2\times1 + 2\times2 + 2\times3 + 3\times1 + 3\times2 + 3\times3}{12}$$

$$= \frac{1+2+3+2+4+6+3+6+9}{36} = \frac{36}{36} = 1$$ (check pmf)

✓

$$f(x,y) = \begin{cases} \frac{xy}{36}, & x=1,2,3, \; y=1,2,3 \\ 0, & \text{otherwise} \end{cases}$$

($\because f(x,y) = \frac{xy}{36} = \frac{x}{6}\cdot\frac{y}{6} = f_X(x) f_Y(y)$ $\therefore$ independent.)

2)

$$f(x,y) = \begin{cases} \frac{1}{6}, & x=1, \; y=1 \\ \frac{2}{6}, & x=2, \; y=2 \\ \frac{3}{6}, & x=3, \; y=3 \\ 0, & \text{otherwise} \end{cases}$$

y \ x	1	2	3	
1	$\frac{1}{6}$	0	0	$\frac{1}{6}$
2	0	$\frac{2}{6}$	0	$\frac{2}{6}$
3	0	0	$\frac{3}{6}$	$\frac{3}{6}$
	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	1

✓

$$\sum_{x\in\{1,2,3\}} \sum_{y\in\{1,2,3\}} f(x,y) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} = 1$$ (check pmf)

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4. A bivariate random vector (X,Y) has the pdf

$$f(x,y)=\begin{cases}3x^{-3}y^{-3}(x^{-2}+y^{-2}-1)^{-\frac{5}{2}} & 0\le x\le 1,\quad 0\le y\le 1\\ 0 & \text{otherwise}\end{cases},$$

1) Derive the marginal pdf $f_X(x)$

2) Derive the cdf $F(x,y)$ for $0\le x\le 1,\quad 0\le y\le 1$.

1)
$$f_X(x) = \int_0^1 3x^{-3}y^{-3}(x^{-2}+y^{-2}-1)^{-\frac{5}{2}}\,dy$$
$$= x^{-3}\int_0^1 3y^{-3}(x^{-2}+y^{-2}-1)^{-\frac{5}{2}}\,dy$$
$$= x^{-3}\left[(x^{-2}+y^{-2}-1)^{-\frac{3}{2}}\right]_0^1$$
$$= x^{-3}\left[x^{3}-0\right] = 1$$

$$\left(\frac{d}{dy}\left[(x^{-2}+y^{-2}-1)^{-\frac{3}{2}}\right] = \left(-\frac{3}{2}\right)\cdot(-2)y^{-3}\left[x^{-2}+y^{-2}-1\right]^{-\frac{5}{2}} = 3y^{-3}(x^{-2}+y^{-2}-1)^{-\frac{5}{2}}\right.$$

Hence,

$$f_X(x) = \begin{cases}1, & 0\le x\le 1\\ 0, & \text{otherwise}\end{cases}$$ □

2)
$$F(x,y) = \int_0^x\int_0^y 3u^{-3}v^{-3}(u^{-2}+v^{-2}-1)^{-\frac{5}{2}}\,dv\,du$$
$$= \int_0^x u^{-3}\left[\int_0^y 3v^{-3}(u^{-2}+v^{-2}-1)^{-\frac{5}{2}}\,dv\right]du$$
$$= \int_0^x u^{-3}\left[(u^{-2}+v^{-2}-1)^{-\frac{3}{2}}\right]_0^y du$$
$$= \int_0^x u^{-3}(u^{-2}+y^{-2}-1)^{-\frac{3}{2}}\,du$$
$$= \left[(u^{-2}+y^{-2}-1)^{-\frac{1}{2}}\right]_0^x$$
$$= (x^{-2}+y^{-2}-1)^{-\frac{1}{2}}$$

$$\left(\frac{d}{du}\left[(u^{-2}+y^{-2}-1)^{-\frac{1}{2}}\right] = \left(-\frac{1}{2}\right)(-2)u^{-3}(u^{-2}+y^{-2}-1)^{-\frac{3}{2}} = u^{-3}(u^{-2}+y^{-2}-1)^{-\frac{3}{2}}\right.$$

Hence,

$$F(x,y) = \begin{cases}(x^{-2}+y^{-2}-1)^{-\frac{1}{2}}, & 0\le x\le 1,\ 0\le y\le 1\\ 0, & \text{otherwise}\end{cases}$$