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High-dimensional data analysis HW1

1. Prove $\boldsymbol{X^T A X} = \sum_{i=1}^{p} \sum_{j=1}^{p} a_{ij} x_i x_j$

Pf:

$$[x_1, x_2, \dots, x_p] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

$$= [x_1, x_2, \dots, x_p] \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p \\ a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pp}x_p \end{bmatrix}$$

$$= (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p)x_1 + (a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p)x_2$$

$$+ (a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pp}x_p)x_p$$

$$= \sum_{i=1}^{p} (a_{i1}x_i + a_{i2}x_i + \cdots + a_{ip}x_p)x_i$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p} (a_{ij}x_j)x_i = \sum_{i=1}^{p} \sum_{j=1}^{p} a_{ij}x_i x_j$$

2. Prove $\frac{\partial \boldsymbol{X^T A X}}{\partial \boldsymbol{X}} = \begin{bmatrix} \frac{\partial \boldsymbol{X^T A X}}{\partial x_1} \\ \vdots \\ \frac{\partial \boldsymbol{X^T A X}}{\partial x_p} \end{bmatrix} = 2\boldsymbol{AX}$

Pf:

From **1.** $\boldsymbol{X^T A X} = \sum_{i=1}^{p} \sum_{j=1}^{p} a_{ij} x_i x_j = \sum_{i=1}^{p} a_{ii} x_i^2 + \sum \sum_{i \neq j} a_{ij}\, x_i x_j$

For $i = 1$

$$\frac{\partial \boldsymbol{X^T A X}}{\partial x_1} = 2a_{11}x_1 + \sum_{j \neq 1} a_{1j} x_j + \sum_{j \neq 1} a_{j1} x_j = 2\sum_{j=1}^{p} a_{1j} x_j$$

Similarly,

$$\frac{\partial \boldsymbol{X^T A X}}{\partial x_i} = 2a_{jj}x_j + \left(\sum_{j=1}^{p} a_{ij} x_j - a_{jj} x_j\right) + \left(\sum_{j=1}^{p} a_{ji} x_j - a_{jj} x_j\right)$$
$$= 2\sum_{j=1}^{p} a_{ij} x_j$$

Therefore,

$$\frac{\partial \boldsymbol{X^T A X}}{\partial \boldsymbol{X}} = \begin{bmatrix} \frac{\partial \boldsymbol{X^T A X}}{\partial x_1} \\ \vdots \\ \frac{\partial \boldsymbol{X^T A X}}{\partial x_p} \end{bmatrix} = \begin{bmatrix} 2\sum_{j=1}^{p} a_{1j} x_j \\ \vdots \\ 2\sum_{j=1}^{p} a_{ij} x_j \end{bmatrix} = 2 \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = 2\boldsymbol{AX}$$