The optional HW in high-dim. class

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Question:

A regression model $\mathbf{y}=\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon}$, where $\mathbf{X}=[\ \mathbf{x}_1\quad \mathbf{x}_2\ ]$; $\mathbf{x}_i=(\ x_{i1}\ x_{i2}\ \cdots\ x_{in}\ )^{\mathrm{T}}, i=1,2$

are standardized with $\sum_{j=1}^{n}x_{ij}=0$ and $\sum_{j=1}^{n}x_{ij}^2=n$. Derive a condition of ρ s.t.

$\|\hat{\boldsymbol{\beta}}^0\|^2\le\|\hat{\boldsymbol{\beta}}\|^2$, where $\hat{\boldsymbol{\beta}}=\dfrac{1}{n-n\rho}\begin{bmatrix}\langle\mathbf{x}_1,\mathbf{y}\rangle-\rho\langle\mathbf{x}_2,\mathbf{y}\rangle\\ \langle\mathbf{x}_2,\mathbf{y}\rangle-\rho\langle\mathbf{x}_1,\mathbf{y}\rangle\end{bmatrix}$, $\hat{\boldsymbol{\beta}}^0=\hat{\boldsymbol{\beta}}|_{\rho=0}=\dfrac{1}{n}\begin{bmatrix}\langle\mathbf{x}_1,\mathbf{y}\rangle\\ \langle\mathbf{x}_2,\mathbf{y}\rangle\end{bmatrix}$ and

$\rho=\dfrac{\langle\mathbf{x}_1,\mathbf{x}_2\rangle}{n}$.

Solution:

$$\|\hat{\boldsymbol{\beta}}^0\|^2=\frac{1}{n^2}\left(\langle\mathbf{x}_1,\mathbf{y}\rangle^2+\langle\mathbf{x}_2,\mathbf{y}\rangle^2\right)$$

$$\|\hat{\boldsymbol{\beta}}\|^2=\frac{1}{n^2(1-\rho^2)^2}\left\{\left(\langle\mathbf{x}_1,\mathbf{y}\rangle-\rho\langle\mathbf{x}_2,\mathbf{y}\rangle\right)^2+\left(\langle\mathbf{x}_2,\mathbf{y}\rangle-\rho\langle\mathbf{x}_1,\mathbf{y}\rangle\right)^2\right\}$$

$$=\frac{1}{n^2(1-\rho^2)^2}\left\{(1+\rho^2)\left(\langle\mathbf{x}_1,\mathbf{y}\rangle^2+\langle\mathbf{x}_2,\mathbf{y}\rangle^2\right)-4\rho\langle\mathbf{x}_2,\mathbf{y}\rangle\langle\mathbf{x}_1,\mathbf{y}\rangle\right\}$$

$$\|\hat{\boldsymbol{\beta}}\|^2-\|\hat{\boldsymbol{\beta}}^0\|^2=\frac{1}{n^2(1-\rho^2)^2}\left\{(1+\rho^2)\left(\langle\mathbf{x}_1,\mathbf{y}\rangle^2+\langle\mathbf{x}_2,\mathbf{y}\rangle^2\right)-4\rho\langle\mathbf{x}_2,\mathbf{y}\rangle\langle\mathbf{x}_1,\mathbf{y}\rangle\right\}-\frac{1}{n^2}\left(\langle\mathbf{x}_1,\mathbf{y}\rangle^2+\langle\mathbf{x}_2,\mathbf{y}\rangle^2\right)$$

$$=\frac{1}{n^2}\left(\langle\mathbf{x}_1,\mathbf{y}\rangle^2+\langle\mathbf{x}_2,\mathbf{y}\rangle^2\right)\left\{\frac{(1+\rho^2)}{(1-\rho^2)^2}-1\right\}-\frac{4\rho}{n^2(1-\rho^2)^2}\langle\mathbf{x}_2,\mathbf{y}\rangle\langle\mathbf{x}_1,\mathbf{y}\rangle\overset{\text{Let}}{\ge}0$$

$$\Rightarrow\left(\langle\mathbf{x}_1,\mathbf{y}\rangle^2+\langle\mathbf{x}_2,\mathbf{y}\rangle^2\right)\{(1+\rho^2)-(1-\rho^2)^2\}-4\rho\langle\mathbf{x}_2,\mathbf{y}\rangle\langle\mathbf{x}_1,\mathbf{y}\rangle\ge0$$

$$\Rightarrow\left(\langle\mathbf{x}_1,\mathbf{y}\rangle^2+\langle\mathbf{x}_2,\mathbf{y}\rangle^2\right)\{3\rho^2-\rho^4\}-4\rho\langle\mathbf{x}_2,\mathbf{y}\rangle\langle\mathbf{x}_1,\mathbf{y}\rangle\ge0$$

$$\Rightarrow\left(\langle\mathbf{x}_1,\mathbf{y}\rangle^2+\langle\mathbf{x}_2,\mathbf{y}\rangle^2\right)(3\rho-\rho^3)-4\langle\mathbf{x}_2,\mathbf{y}\rangle\langle\mathbf{x}_1,\mathbf{y}\rangle\ge0$$

$$\Rightarrow(3\rho-\rho^3)\ge\frac{4\langle\mathbf{x}_2,\mathbf{y}\rangle\langle\mathbf{x}_1,\mathbf{y}\rangle}{\left(\langle\mathbf{x}_1,\mathbf{y}\rangle^2+\langle\mathbf{x}_2,\mathbf{y}\rangle^2\right)}$$

$$\Rightarrow\rho^3-3\rho+\frac{4\langle\mathbf{x}_2,\mathbf{y}\rangle\langle\mathbf{x}_1,\mathbf{y}\rangle}{\left(\langle\mathbf{x}_1,\mathbf{y}\rangle^2+\langle\mathbf{x}_2,\mathbf{y}\rangle^2\right)}\le0$$

The roots of this cubic function are:

$$a = 2\cos\left(\frac{\cos^{-1}(-d/2)}{3}\right), b = 2\cos\left(\frac{\cos^{-1}(-d/2)+2\pi}{3}\right), c = 2\cos\left(\frac{\cos^{-1}(-d/2)-2\pi}{3}\right)$$

where $d = \frac{4\langle \mathbf{x}_2, \mathbf{y}\rangle\langle \mathbf{x}_1, \mathbf{y}\rangle}{\left(\langle \mathbf{x}_1, \mathbf{y}\rangle^2 + \langle \mathbf{x}_2, \mathbf{y}\rangle^2\right)}$.

Thus, if the order of a,b,c is $a > b > c$, the condition of ρ s.t. $\| \hat{\boldsymbol{\beta}}^0 \|^2 \le \| \hat{\boldsymbol{\beta}} \|^2$ is the intersection of $\{-1 < \rho < 1\}$ and $\{\rho < c \ or \ b < \rho < a\}$.