High-dimensional data analysis HW

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Q1. Derive basis function $\phi(x)$ for the polynomial with $x \in \mathbb{R}^3$ or $x \in \mathbb{R}^4$.

Solution.

① $X \in \mathbb{R}^3$, say $X = (X_1, X_2, X_3)$.

Let $\phi(X) = (1, X_1, X_2, X_3, X_1^2, X_2^2, X_3^2, X_1X_2, X_1X_3, X_2X_3)$.
Then, we have a quadratic model,

$$f(X) = w^T \phi(X) \quad \text{with } \phi(X): \mathbb{R}^3 \to \mathbb{R}^{10} ; \; w \in \mathbb{R}^{10}.$$

② $X \in \mathbb{R}^4$, say $X = (X_1, X_2, X_3, X_4)$

Let $\phi(X) = (1, X_1, X_2, X_3, X_4, X_1^2, X_2^2, X_3^2, X_4^2, X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_2X_4, X_3X_4)$
which is a 2-order polynomial basis, $\phi(\cdot): \mathbb{R}^4 \to \mathbb{R}^{15}$.

Q2. Write down the exact formula of $P_\theta(Y|X)$, $\theta = (\mu, \sigma^2)$.

Sol. $P_\theta(Y|X)$ is some probability distribution function, specially in Gaussian, $P_\theta(y|X) = f_{\mu(x), \sigma^2(x)}(y)$

$$= \frac{1}{\sqrt{2\pi\sigma^2(x)}} e^{-\frac{\{y-\mu(x)\}^2}{2\sigma^2(x)}}$$

Q3. Piecewise linear bases

$h_1(X) = I(X < \xi_1)$, $h_4(X) = h_1(X)X$
$h_2(X) = I(\xi_1 \le X < \xi_2)$, $h_5(X) = h_2(X)X$
$h_3(X) = I(\xi_2 \le X < \xi_3)$, $h_6(X) = h_3(X)X$.

Data (Y_i, X_i) $i = 1, \cdots, n$.

$f(X) = \sum_{m=1}^{6} \beta_m h_m(X)$. Derive the LSE $\hat{\beta}$.

$$X^TX = \underbrace{\begin{bmatrix} 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ X_1 \cdots X_{n_1} & 0 \cdots 0 & 0 \cdots 0 \\ 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \\ 0 \cdots 0 & X_{n_1+1} \cdots X_{n_1+n_2} & 0 \cdots 0 \\ 0 \cdots 0 & 0 \cdots 0 & 1 \cdots 1 \\ 0 \cdots 0 & 0 \cdots 0 & X_{n_1+n_2+1} \cdots X_n \end{bmatrix}}_{6 \times n} \underbrace{\begin{bmatrix} 1 & X_1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & X_{n_1+1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & X_{n_1+n_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & X_{n_1+n_2+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & X_n \end{bmatrix}}_{n \times 6}$$

$$= \underbrace{\begin{bmatrix} n_1 & \sum_{i=1}^{n_1} X_i & 0 & 0 & 0 & 0 \\ \sum_{i=1}^{n_1} X_i & \sum_{i=1}^{n_1} X_i^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_2 & \sum_{i=n_1+1}^{n_1+n_2} X_i & 0 & 0 \\ 0 & 0 & \sum_{i=n_1+1}^{n_1+n_2} X_i & \sum_{i=n_1+1}^{n_1+n_2} X_i^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & n-n_1-n_2 & \sum_{i=n_1+n_2+1}^{n} X_i \\ 0 & 0 & 0 & 0 & \sum_{i=n_1+n_2+1}^{n} X_i & \sum_{i=n_1+n_2+1}^{n} X_i \end{bmatrix}}_{6 \times 6}$$

$$X^Ty = \begin{bmatrix} \sum_{i=1}^{n_1} y_i \\ \sum_{i=1}^{n_1} x_i y_i \\ \sum_{i=n_1+1}^{n_2} y_i \\ \sum_{i=n_1+1}^{n_2} x_i y_i \\ \sum_{i=n_1+n_2+1}^{n} y_i \\ \sum_{i=n_1+n_2+1}^{n} x_i y_i \end{bmatrix}$$

.. Here, I provid another notation:

$X_1 = (X_{1i})$; $i = 1, \cdots, n_1$ and $X_1 = (x_1, x_2, \cdots, x_{n_1})$

$X_2 = (X_{2i})$; $i = 1, \cdots, n_2$ and $X_2 = (x_{n_1+1}, \cdots, x_{n_1+n_2})$

$X_3 = (X_{3i})$; $i = 1, \cdots, n-n_1-n_2$ and $X_3 = (x_{n_1+n_2+1}, \cdots, x_n)$

$(y_i)_{i=1,\cdots,n} = \{(y_{1j}), (y_{2k}), (y_{3l})\} = (Y_1, Y_2, Y_3)$

$j = 1, \cdots, n_1$; $k = 1, \cdots, n_2$; $l = 1, \cdots, n-n_1-n_2$.

$$\Rightarrow X^TX = \begin{bmatrix} n_1 & \sum_{i=1}^{n_1} x_{1i} & 0 & 0 & 0 & 0 \\ \sum_{i=1}^{n_1} x_{1i} & \sum_{i=1}^{n_1} x_{1i}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_2 & \sum_{i=1}^{n_2} x_{2i} & 0 & 0 \\ 0 & 0 & \sum_{i=1}^{n_2} x_{2i} & \sum_{i=1}^{n_2} x_{2i}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & n-n_1-n_2 & \sum_{i=1}^{n-n_1-n_2} x_{3i} \\ 0 & 0 & 0 & 0 & \sum_{i=1}^{n-n_1-n_2} x_{3i} & \sum_{i=1}^{n-n_1-n_2} x_{3i}^2 \end{bmatrix}; \quad X^Ty = \begin{bmatrix} n_1 \bar{y}_{1\cdot} \\ \langle X_1, Y_1 \rangle \\ n_2 \bar{y}_{2\cdot} \\ \langle X_2, Y_2 \rangle \\ (n-n_1-n_2)\bar{y}_{3\cdot} \\ \langle X_3, Y_3 \rangle \end{bmatrix}$$

$$\Rightarrow (X^TX)^{-1} = \begin{bmatrix} \frac{1}{n_1 SS_{X_1}} \begin{pmatrix} \sum_{i=1}^{n_1} x_{1i}^2 & -\sum_{i=1}^{n_1} x_{1i} \\ -\sum_{i=1}^{n_1} x_{1i} & n_1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \frac{1}{n_2 SS_{X_2}} \begin{pmatrix} \sum_{i=1}^{n_2} x_{2i}^2 & -\sum_{i=1}^{n_2} x_{2i} \\ -\sum_{i=1}^{n_2} x_{2i} & n_2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \frac{1}{(n-n_1-n_2) SS_{X_3}} \begin{pmatrix} \sum_{i=1}^{n-n_1-n_2} x_{3i}^2 & -\sum_{i=1}^{n-n_1-n_2} x_{3i} \\ -\sum_{i=1}^{n-n_1-n_2} x_{3i} & n-n_1-n_2 \end{pmatrix} \end{bmatrix}, \quad \text{let } n_3 = n-n_1-n_2$$

$$\Rightarrow \hat{\beta} = (X'X)^{-1}X'y = \begin{bmatrix} \frac{1}{n_1 SS_{X_1}} \left(\sum_{i=1}^{n_1} x_{1i}^2 \sum_{i=1}^{n_1} y_{1i} - \sum_{i=1}^{n_1} x_{1i} \sum_{i=1}^{n_1} x_{1i} y_{1i} \right) \\ \frac{1}{n_1 SS_{X_1}} \left(n_1 \sum_{i=1}^{n_1} x_{1i} y_{1i} - \sum_{i=1}^{n_1} x_{1i} \sum_{i=1}^{n_1} y_{1i} \right) = \frac{SS_{X_1 Y_1}}{SS_{X_1}} \\ \frac{1}{n_2 SS_{X_2}} \left(\sum_{i=1}^{n_2} x_{2i}^2 \sum_{i=1}^{n_2} y_{2i} - \sum_{i=1}^{n_1} x_{2i} \sum_{i=1}^{n_2} x_{2i} y_{2i} \right) \\ \frac{1}{n_2 SS_{X_2}} \left(n_2 \sum_{i=1}^{n_2} x_{2i} y_{2i} - \sum_{i=1}^{n_2} x_{2i} \sum_{i=1}^{n_2} y_{2i} \right) = \frac{SS_{X_2 Y_2}}{SS_{X_2}} \\ \frac{1}{n_3 SS_{X_3}} \left(\sum_{i=1}^{n_3} x_{3i}^2 \sum_{i=1}^{n_3} y_{3i} - \sum_{i=1}^{n_3} x_{3i} \sum_{i=1}^{n_3} x_{3i} y_{3i} \right) \\ \frac{1}{n_3 SS_{X_3}} \left(n_3 \sum_{i=1}^{n_3} x_{3i} y_{3i} - \sum_{i=1}^{n_3} x_{3i} \sum_{i=1}^{n_3} y_{3i} \right) = \frac{SS_{X_3 Y_3}}{SS_{X_3}} \end{bmatrix}$$