Advanced Probability I, 2013 Fall, Midterm Exam

Name:

1. Prove or disprove the following.

(i) If X and Y are random variable on a measurable space $(\Omega, \mathbf{F})$, then $aX + bY$ is a random variable for $a, b \in R$.

(ii) If X_n, $n = 1, 2, ...$ are sequence of random variables a measurable space $(\Omega, \mathbf{F})$, then $\sup_n X_n$ is a random variable.

2. Let $(R^i, \Re^i)$ and $(R^k, \Re^k)$ be Borel measurable spaces on i and k dimensional Euclid spaces, respectively. Prove the following:

If $f \ : R^i \to R^k$ is continuous, then $f \ : (R^i, \Re^i) \to (R^k, \Re^k)$ is measureable.

3. Let $X_n, \quad n = 1, 2, ...$ be a sequence of random variables, and $a_n, \quad n = 1, 2, ...$ be a sequence of constants. Suppose $\sum_n P\{|X_n| > a_n\} < \infty$.

(i) Derive $P(\Xi)$, where $\Xi = \bigcup_n \bigcap_{k=n}^{\infty} \{|X_k| \le a_k\}$.

(ii) Derive $P\left(\sum_n |X_n| < \infty\right)$ if $\sum_{n=1}^{\infty} a_n < \infty$.

4. State and prove the second Borel-Cantelli lemma

5. Let $(\Omega, \mathbf{F}, P)$ be a probability space. Prove the following:

(i) Let $X : \Omega \to R$ be a random variable. Then, $\sigma(X) = \{ X^{-1}(B); B \in \Re \}$ is a sigma-field.

(ii) Let $(X,Y) : \Omega \to R^2$ be a random vector.

Then, $\sigma(X,Y) = \{ (X,Y)^{-1}(B); B \in \Re^2 \}$ is a sigma-field and $\sigma(X) \subset \sigma(X,Y)$.

6. Let $(\Omega, \mathbf{F}, \mu)$ be a measure space. The μ*-completion* of $\mathbf{F}$ is the collection of all $E \subset \Omega$ for which there exists $A, B \in \mathbf{F}$ with $A \subset E \subset B$ and $\mu(B - A) = 0$. That is, the μ-completion of $\mathbf{F}$ is

$$\mathbf{F}^* = \{ E \subset \Omega; \quad A \subset E \subset B, \quad \mu(B - A) = 0 \text{ for some } A, B \in \mathbf{F} \}.$$

Show that $\mathbf{F}^*$ is a σ-field.

7. Let $\Omega = \{-N,..,-2,-1,0,1,2,...,N\}$, $\mathbf{F} = 2^{\Omega}$, and ν be the counting measure [$\nu(A) = \# A, \quad \forall A \in \mathbf{F}$] on $\mathbf{F}$. Let $f(\omega) = \sum_{k=-N}^{N} k I_{\{k\}}(\omega)$.

1) Calculate $\int f d\nu$.

2) Calculate $\int | f | d\nu$.

8. (i) State and prove Fatou's Lemma.

(ii) Give an example that the interchange of integral and limit cannot change but Fatou's Lemma is still valid.

NOTE: This is my simplified answers. You need to write more detailed calculations.

Answer 2.

Let $B \in \Re^k$. Since the Borel sigma-field is generated by the open sets, there exist a collection of open sets $\{O_\alpha, \alpha \in \mathrm{A}\}$ such that $B = \bigcup_{\alpha \in \mathrm{A}} O_\alpha$. By the continuity of $f : R^i \to R^k$, $f^{-1}(O_\alpha) \in \Re^i$ for $\alpha \in \mathrm{A}$. It follows that $f^{-1}(B) = f^{-1}(\bigcup_\alpha O_\alpha) = \bigcup_\alpha f^{-1}(O_\alpha) \in \Re^i$. Therefore, f is Borel measurable.

Answer 3.

(i) By the first Borel-Cantelli's Lemma, if $\sum_n P(A_n) < \infty$, then $P(\limsup A_n) = 0$.

Thus, $P(\Xi) = P(\liminf\{|X_n| \le a_n\}) = 1 - P(\limsup\{|X_n| > a_n\}) = 1$.

(ii) $P\left(\sum_n |X_n| < \infty\right) = 1$ since

$$1 = P\left(\bigcup_n \bigcap_{k=n}^{\infty} \{|X_k| \le a_k\}\right)$$

$$= \{\exists n \in \mathbf{N}, \forall k \ge n, \quad |X_k| \le a_k\} \subset \{\exists n \in \mathbf{N}, \sum_{k \ge n} |X_k| \le \sum_{k \ge n} a_k < \infty\} \subset \{\sum_n |X_n| < \infty\}$$

Answer 6.

(i) Let $E = \Omega$ and $A = B = \Omega$.

(ii) If $E \in \mathbf{F}^*$, there exist $A, B \in \mathbf{F}$ with $A \subset E \subset B$ and $\mu(B - A) = 0$. Then, $A^c \supset E \supset B^c$ with $\mu(A^c - B^c) = \mu(A^c \cap B) = \mu(B - A) = 0$ and $A^c, B^c \in \mathbf{F}$.

(iii) if $E_i \in \mathbf{F}^*$, the there exist $A_i, B_i \in \mathbf{F}$ with $A_i \subset E_i \subset B_i$ and $\mu(B_i - A_i) = 0$. Then, $\bigcup_i A_i \subset \bigcup_i E_i \subset \bigcup_i B_i$ and

$$\mu\left(\bigcup_i B_i - \bigcup_i A_i\right) = \mu\left(\bigcup_i (B_i - A_i)\right) \le \sum_i \mu(B_i - A_i) = 0.$$

Answer 7.

$$\int f d\nu = \sum_{k=-N}^{N} \int k I_{\{k\}}(\omega) d\nu(\omega) = -\sum_{k=1}^{N} k + \sum_{k=1}^{N} k = 0.$$

$$\int |f| d\nu = \sum_{k=-N}^{N} \int |k| I_{\{k\}}(\omega) d\nu(\omega) = \sum_{k=1}^{N} k + \sum_{k=1}^{N} k = N(N+1).$$