Advanced Probability I, 2013 Fall, Homework #2

1. Presentation
   1) About intersection/union of sigma-field (already finished)
   2) Suppose X and Y are random variables. Show that $|X\text{-}Y|$ and $\max(X,Y)$ are also random variables (some details are necessary).
      Give a sufficient condition that X/Y is random variable.
   3) $T^{-1}(\bigcup_n A_n) = \bigcup_n T^{-1}(A_n)$ as mentioned in class.

2. Let $X : (\Omega, \mathbf{F}, P) \to (R^1, \Re^1)$ be a random variable, representing the failure time of an electric component.

(i) Suppose an engineer *cannot* observe the exact failure time X. Instead, he/she can observe the failure status of the electric component at a fix time t>0. That is, the engineer can identify whether the event $\{ X \le t \} = \{ \omega : X(\omega) \le t \}$ occurs or not. Show that $\{ X \le t \} \in \mathbf{F}$.

(ii) Find $\sigma(\{X \le t\})$, a σ-field generated by the event $\{ X \le t \}$.

(iii) Suppose that an engineer can observe the failure status of the electric component at two different times $t_1 < t_2$. Find $\sigma(\{ X \le t_1 \}, \{ X \le t_2 \})$.

(iv) Suppose that an engineer can observe the failure status of the electric component at $t_1 < t_2 < \cdots < t_k$. Explicitly write

$\mathbf{F}_k = \sigma([\{ X \le t_j \}; j = 1,2,...,k])$. How many elements are there?

($\mathbf{F}_j$ represents all events that the engineer can observe before t_k)

(v) Show that $\mathbf{F}_{i-1} \subset \mathbf{F}_i$ for $i = 2, ..., k$.

(vi) Prove or disprove $\{ X > t_i \} \in \mathbf{F}_i$, $\{ X > t_i \} \in \mathbf{F}_{i-1}$, and $\{ X \ge t_i \} \in \mathbf{F}_{i-1}$.

NOTE: This is my simplified answers. You need to write more detailed calculations.

Answer 2

(ii) $\sigma(\{X \le t\}) = [\phi, \{X \le t\}, \{X > t\}, \Omega]$.

(iii) $$\begin{aligned}&\sigma(\{X \le t_1\}, \{X \le t_2\}) \\ &= [\phi, \{X \le t_1\}, \{X \le t_2\}, \{X > t_1\}, \{X > t_2\}, \{t_1 < X \le t_2\}, \{X \le t_1 \text{ or } X > t_2\}, \Omega]\end{aligned}.$$

8 elements.

(iv) Let $t_0 = -\infty$. Then,

$$\begin{aligned}\mathbf{F}_k &= \sigma([\{X \le t_j\};\, j = 1,2,\ldots,k]) \\ &= \left\{ \bigcup_{j=1}^{k+1} A_j \,\middle|\, A_j = \{X \in (t_{j-1}, t_j]\} \text{ or } \phi,\ j = 1,\ldots,k; \quad A_{k+1} = \{X > t_k\} \text{ or } \phi \right\}\end{aligned}$$

Hence, the number of elements is 2^{k+1}. I check this number by a special case of k=3 as follows:

$$\begin{aligned}&\sigma(\{X \le t_1\}, \{X \le t_2\}, \{X \le t_3\}) = \\ &[\phi, \Omega, \\ &\{X \le t_1\}, \{X \le t_2\}, \{X \le t_3\}, \\ &\{X > t_1\}, \{X > t_2\}, \{X > t_3\} \\ &\{t_1 < X \le t_2\}, \{t_2 < X \le t_3\}, \{t_2 < X \le t_3\}, \{t_{`} < X \le t_3\}, \\ &\{t_1 < X \le t_2\}^c, \{t_2 < X \le t_3\}^c, \{t_2 < X \le t_3\}^c, \{t_{`} < X \le t_3\}^c]\end{aligned}$$

Hence, we have 16= 2^{3+1} elements.

(v) Omit [it follows from (A1)].

(vi) From (A1), $\{X > t_i\} \in \mathbf{F}_i$, $\{X > t_i\} \notin \mathbf{F}_{i-1}$ and $\{X \ge t_i\} = \{X > t_{i-1}\} \in \mathbf{F}_{i-1}$.