**Advanced Probability I, 2013 Fall, Homework #1 (due 10/3) [+4]**

Q1 [+2] Presentation:

1) What is the definition of countable set?
2) The proofs for Exercise 1.9 and 1.11 (Casella & Berger, Statistical inference) (within 40 minutes).

Q2 [+2] Let $\mathbf{B} = \sigma(\mathbf{B}_0)$ be the Borel $\sigma$-field on $\Omega = (0,1]$.

1) $\{x\} \in \mathbf{B}$ for any $x \in \Omega$
2) $C \in \mathbf{B}$ for any countable subset $C \subset \Omega$
3) $G \in \mathbf{B}$ for any open interval $G = (a,b)$ on $(0,1]$
4) $F \in \mathbf{B}$ for any closed interval $F = [a,b]$ on $(0,1]$
5) $G \in \mathbf{B}$ for any open set $G$ on $(0,1]$
6) $F \in \mathbf{B}$ for any closed set $F$ on $(0,1]$
7) Find a set $H \subset \Omega$ that does not belong to $\mathbf{B}$.

Advanced Probability I, Homework #1

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1) Let $A_n = \left(x - \frac{1}{n}\ ,\ x\right] \in B_0$, for $n \in \mathbb{N},\ x \in \Omega$. Then $A_n$ is decreasing. Hence, the intersection of $A_n$, $\bigcap_n A_n = \{x\}$. Since $\bigcap_n A_n \in B$, $\{x\} \in B$.

2) Let $A_n = \bigcup_k (a_k - \frac{1}{n}\ , a_k] \in B_0$, for $n, k \in \mathbb{N}$, $a_k \in \Omega$ for all $k$. Then $A_n$ is decreasing. Hence, the intersection of $A_n$, $\bigcap_n A_n = \{a_1, a_2, \ldots\}$. Since $k \in \mathbb{N}$, $\{a_1, a_2, \ldots\}$ is a countably infinite and $\{a_1, a_2, \ldots\} \in B$.

3) Let $A_n = \left(a + \frac{1}{n}\ ,\ b - \frac{1}{n}\right] \in B_0$ for some $a, b \in \Omega$. Then $A_n$ is increasing. Hence, let $G = \bigcup_n A_n = (a, b) \in B$.

4) Let $A_n = \left(a - \frac{1}{n}, b\right] \in B_0$ for some $a, b \in \Omega$. Then $A_n$ is decreasing. Hence, let $F = \bigcap_n A_n = [a, b] \in B$.

5) If $G$ is open on $(0, 1]$ and $x \in \Omega$. There exist rationals $a_x, b_x \in \Omega$. such that $x \in (a_x, b_x] \subset G$. Then $G = \bigcup_{x \in G} (a_x, b_x]$ and there only countably many intervals with rational endpoints. Thus, $G \in B$.

6) By 5), since $G$ is an open set and $G \in B$, $C = G^c$ is a closed set. Thus, $F \in B$ by the definition of $\sigma$-field.

7) Let $x \in H$ be the element of $H \subset \Omega$ which is a set of the irrational numbers and $x$ can be expressed as following :

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

where $a_k$ is any positive number for all $k \in \mathbb{N}$. In this case, $H$ is not a Borel set and $H \notin B$.