Stat Papers (2019) 60:1101–1118
https://doi.org/10.1007/s00362-016-0865-5

REGULAR ARTICLE

Bivariate dependence measures and bivariate competing risks models under the generalized FGM copula

Jia-Han Shih[1] · **Takeshi Emura**[1]

Received: 7 May 2016 / Revised: 4 December 2016 / Published online: 22 December 2016

Abstract The first part of this paper reviews the properties of bivariate dependence measures (Spearman's rho, Kendall's tau, Kochar and Gupta's dependence measure, and Blest's coefficient) under the generalized Farlie–Gumbel–Morgenstern (FGM) copula. We give a few remarks on the relationship among the bivariate dependence measures, derive Blest's coefficient, and suggest simplifying the previously obtained expression of Kochar and Gupta's dependence measure. The second part of this paper derives some useful measures for analyzing bivariate competing risks models under the generalized FGM copula. We obtain the expression of sub-distribution functions under the generalized FGM copula, which has not been discussed in the literature. With the Burr III margins, we show that our expression has a closed form and generalizes the reliability measure previously obtained by Domma and Giordano (Stat Pap 54(3):807–826, 2013).

Keywords Blest's coefficient · Competing risk · FGM copula · Kendall's tau · Spearman's rho

Electronic supplementary material The online version of this article (doi:10.1007/s00362-016-0865-5) contains supplementary material, which is available to authorized users.

✉ Takeshi Emura
takeshiemura@gmail.com; emura@stat.ncu.edu.tw

Jia-Han Shih
tommy355097@gmail.com

[1] Graduate Institute of Statistics, National Central University, No. 300, Zhongda Rd., Zhongli District, Taoyuan City 32001, Taiwan

1 Introduction

In a bivariate distribution, studying dependence measures between two random variables is essential. While the Pearson correlation may be the most popular measure for dependence, it is affected by the marginal distributions and is meaningful only for continuous variables.

Bivariate dependence measures that are free from the marginal distributions include, among others, Spearman's rho, Kendall's tau, Blest's coefficient, and Kochar and Gupta's dependence measure. Spearman's rho and Kendall's tau are based on the concept of concordance and they are related to each other (Capéraà and Genest 1993; Nelsen 2006). Blest's coefficient (Blest 2000) is derived by a graphical representation of rank differences. The dependence measure of Kochar and Gupta (1987) is based on the concept of quadrant dependence (Lehmann 1966). These dependence measures are free from the marginal distributions; they depend only on the copula between two continuous random variables (Scarsini 1984; Nelsen 2006). Spearman's rho and Kendall's tau are appropriately defined even for non-continuous variables (Nešlehová 2007).

A bivariate copula is a bivariate distribution function of two uniform random variables on the unit interval [0, 1] (Nelsen 2006). By using copulas, one can model the dependence structure between two random variables having arbitrary marginal distributions. Well-known copulas include the Clayton (Clayton 1978), Frank (Frank 1979; Genest 1987), Gumbel (Gumbel 1960a) and Joe (Joe 1993) copulas from the Archimedean family and the Gaussian and $t-$copulas from the elliptical family.

This paper focuses on the Farlie–Gumbel–Morgenstern (FGM) copula and the generalized FGM copula. The FGM copula is a copula corresponding to the FGM distribution introduced by Morgenstern (1956), which is also traced back to Eyraud (1936). In application to the diabetic retinopathy study, Louzada et al. (2013) demonstrated that, among a pool of copulas, the FGM copula yields the best fit for the data.

The FGM distribution is defined as a bivariate distribution function $F(x, y) = F_1(x)F_2(y)[1 + \theta\{1 - F_1(x)\}\{1 - F_2(y)\}]$ with marginal distribution functions F_1 and F_2 and a parameter θ. Gumbel (1960b) studied the FGM distribution with exponential margins. For certain functions $A(\cdot)$ and $B(\cdot)$, Farlie (1960) generalized the FGM distribution by $F(x, y) = F_1(x)F_2(y)[1+\theta\{1-A(F_1(x))\}\{1-B(F_2(y))\}]$. Due to its nice mathematical properties, the generalized FGM copula has been studied by many authors, including Bairamov and Kotz (2002), Nelsen (2006), Amini et al. (2011), Domma and Giordano (2013, 2016), to name but a few. While there exist a few different ways of generalizing the FGM copula (e.g., Rodíguez-Lallena and Úbeda-Flores 2004; Amblard and Girard 2009), we focus on the particular generalization considered by Bairamov and Kotz (2002).

The first part of this paper reviews the properties of selected dependence measures (Spearman's rho, Kendall's tau, Blest's coefficient, Kochar and Gupta's dependence measure) under the generalized FGM copula of Bairamov and Kotz (2002). We give a few remarks on the relationship among the dependence measures, derive Blest's coefficient, and suggest simplifying the expression of Kochar and Gupta's dependence measure previously obtained by Amini et al. (2011).

The second yet more novel contribution of this paper is to derive the expressions of sub-distribution functions with competing risks under the generalized FGM copula. Such expressions have not been derived in the literature, though the sub-distribution functions play a fundamental role in competing risks models (Gray 1988; Crowder 2001; Bakoyannis and Touloumi 2012). With the Burr III margins, we show that our expression is explicitly written and is a generalization of the reliability measure obtained by Domma and Giordano (2013).

This paper is organized as follows. Section 2 introduces copulas and basic notations. Section 3 reviews the generalized FGM copula and its dependence measures. Section 4 derives the expressions of sub-distribution functions with dependent competing risks models under the generalized FGM copula. Section 5 illustrates the usage of the generalized FGM copula by a real dataset. Section 6 concludes the paper. Supplementary Material includes detailed proofs and simulation results to support the results.

2 Copula

A bivariate copula is a bivariate distribution function whose margins are uniform on the unit interval [0, 1] (Nelsen 2006). A bivariate copula is a map $C : [0, 1]^2 \rightarrow [0, 1]$ with $C(u, 1) = u$ and $C(1, v) = v$. Let X and Y be random variables with a joint distribution function $F(x, y) = \Pr(X \leq x, Y \leq y)$ and continuous marginal distribution functions $F_1(x) = \Pr(X \leq x)$ and $F_2(y) = \Pr(Y \leq y)$, respectively. By Sklar's theorem (Sklar 1959), one has a unique representation

$$F(x, y) = C\{F_1(x), F_2(y)\}.$$

This representation is extremely useful as it separates the dependence structure C from the marginal distributions F_1 and F_2. Many dependence measures between X and Y, including Spearman's rho and Kendall's tau, depend only on the copula function $C(\cdot, \cdot)$.

Spearman's rho (ρ) and Kendall's tau (τ) are expressed as

$$\rho = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3, \quad \tau = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1.$$

They are interpreted similarly to the Pearson correlation; the possible ranges are $\rho \in [-1, 1]$ and $\tau \in [-1, 1]$; the independence implies $\rho = \tau = 0$.

The one-parameter FGM copula is $C(u, v) = uv\{1 + \theta(1 - u)(1 - v)\}$, where $\theta \in [-1, 1]$ is a parameter. The copula density is $c(u, v) = 1 + \theta(1 - 2u)(1 - 2v)$. Spearman's rho and Kendall's tau are $\theta/3$ and $2\theta/9$, respectively (Schucany et al. 1978; Nelsen 2006). Therefore, the range of Spearman's rho is $[-1/3, 1/3]$ and the range of Kendall's tau is $[-2/9, 2/9]$. The advantage of the FGM copula is the simple linear expressions of ρ and τ. The disadvantage is narrow ranges for ρ and τ.

The FGM copula belongs to neither the Archimedean nor elliptical family when $\theta \neq 0$ and it reduce to the independent copula when $\theta = 0$. However, it has been widely discussed in the literature due to its mathematically interesting properties (Nelsen 2006).

3 Generalized Farlie–Gumbel–Morgenstern copula

The generalized FGM distribution is proposed by Bairamov and Kotz (2002) to extend the narrow range of Spearman's rho (ρ) and Kendall's tau (τ). With additional parameters, $p, q \geq 1$, the generalized FGM copula is

$$C(u, v) = uv\{1 + \theta(1 - u^p)^q(1 - v^p)^q\}, \tag{1}$$

where the possible range of θ is

$$-\min\left\{1, \frac{1}{p^{2q}}\left(\frac{1+pq}{q-1}\right)^{2q-2}\right\} \leq \theta \leq \frac{1}{p^q}\left(\frac{1+pq}{q-1}\right)^{q-1}.$$

The copula density is $c(u, v) = 1 + \theta(1 - u^p)^{q-1}\{1 - (1 + pq)u^p\}(1 - v^p)^{q-1}\{1 - (1 + pq)v^p\}$. The case of $p = q = 1$ corresponds the original FGM copula.

3.1 Spearman's rho and Kendall's tau

We review the expressions for Spearman's rho and Kendall's tau, and then we discuss their properties.

Proposition 1 (Spearman's rho and Kendall's tau) *Under the generalized FGM copula in Eq. (1), Spearman's rho* (ρ) *and Kendall's tau*(τ) *are*

$$\rho = 12\left\{\frac{q}{2+pq}B\left(\frac{2}{p}, q\right)\right\}^2\theta, \quad \tau = 8\left\{\frac{q}{2+pq}B\left(\frac{2}{p}, q\right)\right\}^2\theta,$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ *is the beta function and* $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$ *is the gamma function. Hence,* $2\rho = 3\tau$ *in the range of* θ.

Proposition 1 is an obvious simplification from the results of Amini et al. (2011) who obtained

$$\rho = \frac{12\theta}{p^2}\left\{B\left(\frac{2}{p}, q+1\right)\right\}^2 \tag{2}$$

and

$$\tau = \frac{4\theta}{p^2}\left\{B\left(\frac{2}{p}, q+1\right)\right\}^2 + \frac{4\theta}{p^2}\left\{B\left(\frac{2}{p}, q\right) - (1+pq)B\left(\frac{2}{p}+1, q\right)\right\}^2$$
$$+\frac{4\theta}{p^2}\left\{B\left(\frac{2}{p}, 2q\right) - (1+pq)B\left(\frac{2}{p}+1, 2q\right)\right\}^2. \tag{3}$$

Proposition 1 is the consequence of applying the properties of the beta function

$$B(a+1,b)=\frac{a}{a+b}B(a,b),\quad B(a,b+1)=\frac{b}{a+b}B(a,b)$$

to Equations (2) and (3). Our independent derivations, without passing through Equations (2) and (3), are given in Supplementary Material. Proposition 1 is also an immediate consequence from more general results of Domma and Giordano (2013, 2016).

Proposition 1 reveals the relationship $2\rho = 3\tau$, where

$$\rho \in \left[-\min\left\{1, \frac{1}{p^{2q}}\left(\frac{1+pq}{q-1}\right)^{2q-2}\right\}12M, \frac{1}{p^q}\left(\frac{1+pq}{q-1}\right)^{q-1}12M\right],$$
$$\tau \in \left[-\min\left\{1, \frac{1}{p^{2q}}\left(\frac{1+pq}{q-1}\right)^{2q-2}\right\}8M, \frac{1}{p^q}\left(\frac{1+pq}{q-1}\right)^{q-1}8M\right],$$

and $M = \{qB(2/p,q)/(2+pq)\}^2$. The relationship $2\rho = 3\tau$ is well-known under the original FGM copula over a limited range, $\rho = [-1/3, 1/3]$ and $\tau \in [-2/9, 2/9]$ (Schucany et al. 1978; Nelsen 2006). Under the generalized FGM copula with parameters $p = 3$ and $q = 2$, for instance, the ranges are extended to $\rho \in [-0.368, 0.473]$ and $\tau \in [-0.245, 0.315]$.

Figure 1 displays the range of Kendall's tau and Spearman's rho. While the ranges of Kendall's tau and Spearman's rho are extended by sensible choices of p and q, the ranges are still narrow.

The relationship $\rho = 1.5\tau$ in Proposition 1 shows that the magnitude of Spearman's rho is 1.5 times larger than that of Kendall's tau. Capéraà and Genest (1993) gave

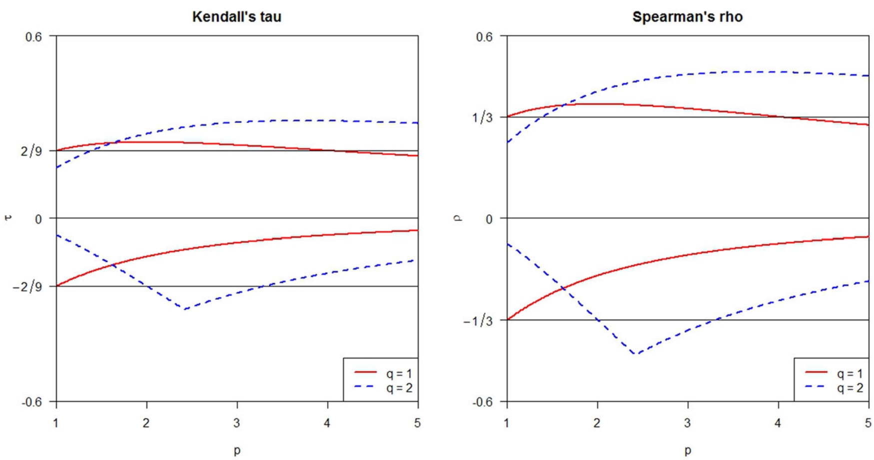

Fig. 1 The range of Kendall's tau (τ) and Spearman's rho (ρ) under the generalized FGM copula with parameters $p \in [1, 5]$ and $q = 1$ or 2

near minimal conditions that make Spearman's rho larger than Kendall's tau, say $\rho \geq \tau \geq 0$. Below, we examine their results with the generalized FGM copula.

Lehmann (1966) and Esary et al. (1967) introduced the following concept of positive dependence for a bivariate random vector (X, Y) with a copula C:

- X is said to be *left-tail decreasing* in Y, denoted as $LTD(X|Y)$, if and only if $\Pr(X \leq x|Y \leq y) = C(u, v)/v$ is a non-increasing function of $v = \Pr(Y \leq y)$ for any $u = \Pr(X \leq x)$.
- X is said to be *right-tail increasing* in Y, denoted as $RTI(X|Y)$, if and only if $\Pr(X \leq x|Y > y) = \{u - C(u, v)\}/(1 - v)$ is a non-increasing function of $v = \Pr(Y \leq y)$ for any $u = \Pr(X \leq x)$.

One can easily check that a pair (X, Y) with the FGM copula satisfies both $LTD(X|Y)$ and $RTI(X|Y)$ for $\theta > 0$. However, we will verify that a pair (X, Y) with the generalized FGM copula is $LTD(X|Y)$, but is not $RTI(X|Y)$ for $\theta > 0$, $p \neq 1$ and $q \neq 1$. Under the generalized FGM copula with $\theta > 0$,

$$\frac{C(u, v)}{v} = u\{1 + \theta(1 - u^p)^q(1 - v^p)^q\}$$

is a non-increasing function of v for all u. Thus, $LTD(X|Y)$ holds true. On the other hand, for $p = q = 2$ and $v' \geq v$, we know

$$\frac{u - C(u, v)}{1 - v} \geq \frac{u - C(u, v')}{1 - v'} \Leftrightarrow \frac{v(1 - v^2)^2}{1 - v} \leq \frac{v'(1 - v'^2)^2}{1 - v'}.$$

The last inequality does not hold for all $v' \geq v$; letting $v = 0.7$ and $v' = 0.8$, we have

$$0.7(1 - 0.7^2)^2/(1 - 0.7) = 0.6069 > 0.5184 = 0.8(1 - 0.8^2)^2/(1 - 0.8),$$

which contradicts to $RTI(X|Y)$.

According to Capéraà and Genest (1993), $LTD(X|Y)$ and $RTI(X|Y)$ implies $\rho \geq \tau \geq 0$, but not vice versa. They exhibited the Mardia copula as an example of copulas that holds $\rho \geq \tau \geq 0$ without satisfying $LTD(X|Y)$ and $RTI(X|Y)$ conditions. The generalized FGM copula gives an additional example as we have demonstrated above.

3.2 Blest's coefficient and symmetrized Blest's coefficient

Blest (2000) proposed a rank correlation measure which emphasizes the differences in the top ranks. This concept differs from Spearman's rank correlation measure that gives the same weight for all ranks. To be specific, Blest's coefficient (ν) is defined as

$$\nu = 2 - 12 \int_0^1 \int_0^1 (1 - u)^2 v dC(u, v).$$

Blest's coefficient is not symmetric unless a copula is symmetric. Therefore, Genest and Plante (2003) suggested a symmetrized version of Blest's coefficient (ξ), defined as

$$\xi = \frac{\nu + \tilde{\nu}}{2} = -4 + 6 \int_0^1 \int_0^1 uv(4 - u - v) dC(u, v),$$

where

$$\tilde{\nu} = 2 - 12 \int_0^1 \int_0^1 (1 - v)^2 u dC(u, v).$$

The following theorem derives the expressions of Blest's coefficient and symmetrized Blest's coefficient based on the definitions above.

Theorem 1 (Blest's coefficient and symmetrized Blest's coefficient) *Under the generalized FGM copula in Eq. (1),*

$$\nu = \xi = 24 \left\{ \frac{q}{2 + pq} B\left(\frac{2}{p}, q\right) \right\}^2 \theta - 24 \left\{ \frac{q}{2 + pq} B\left(\frac{2}{p}, q\right) \right\} \left\{ \frac{q}{3 + pq} B\left(\frac{3}{p}, q\right) \right\} \theta.$$

Proof of Theorem 1 Since the generalized FGM copula in Eq. (1) is symmetric in its margins, we have $\nu = \tilde{\nu} = \xi$. According to Genest and Plante (2003), an alternative representation of Blest's coefficient is

$$\nu = -2 + 24 \int_0^1 \int_0^1 (1 - u) C(u, v) du dv.$$

Then the results can be easily obtained from straightforward calculations. □

Corollary 1 *The relationship between Blest's coefficient and Spearman's rho is*

$$\nu = 2\rho \left\{ 1 - \frac{(2 + pq)\Gamma(3/p)\Gamma(2/q + q)}{(3 + pq)\Gamma(3/p + q)\Gamma(2/p)} \right\}.$$

Proof of Corollary 1 According to Theorem 1, we have

$$\nu = 2\rho \left\{ 1 - \frac{q/(3 + pq) B(3/p, q)}{q/(2 + pq) B(2/p, q)} \right\} = 2\rho \left\{ 1 - \frac{(2 + pq)\Gamma(3/p)\Gamma(2/q + q)}{(3 + pq)\Gamma(3/p + q)\Gamma(2/p)} \right\}.$$

Thus, we obtain the desired result. □

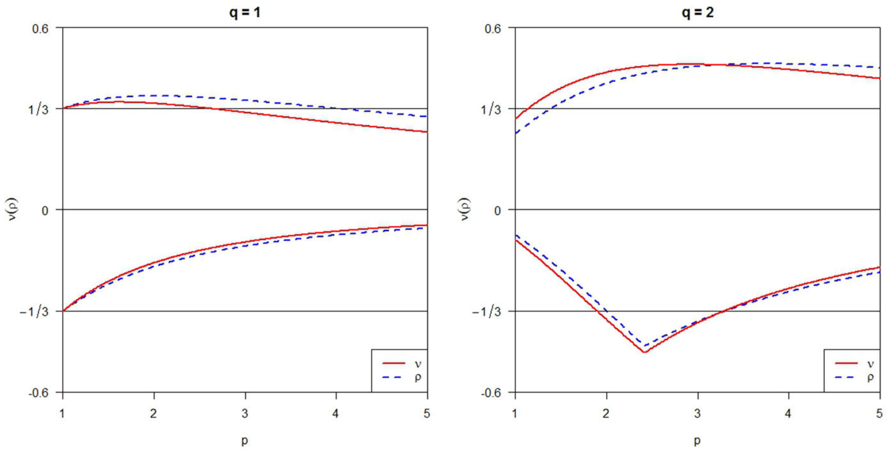

Fig. 2 The range of Blest's coefficient (ν) and Spearman's rho (ρ) under the generalized FGM copula with parameters $p \in [1, 5]$ and $q = 1$ or 2

For instance, the relationship between Blest's coefficient and Spearman's rho are

$$\nu = \frac{2(p+5)}{3(p+3)}\rho, \quad \text{for} \quad q = 1,$$
$$\nu = \frac{2(2p^2+15p+19)}{3(p+3)(2p+3)}\rho, \quad \text{for} \quad q = 2,$$

Figure 2 compares the range of Blest's coefficient and Spearman's rho. For the case $q = 1$, the range of Blest's coefficient is smaller or equal to the range Spearman's rho since $2(p+5)/3(p+3)$ is decreasing in p. The equality $\nu = \rho = \theta/3$ occurs at $p = q = 1$ corresponding to the original FGM copula [see also, Example 1 of Genest and Plante (2003)]. Figure 2 shows that with sensible choices of p and q, the range of Blest's coefficient can be slightly larger than the range of Spearman's rho.

3.3 Kochar and Gupta's dependence measure

The dependence measure of Kochar and Gupta (Kochar and Gupta 1987) is based on the concept of quadrant dependence. A bivariate random vector (X, Y) is positive quadrant dependent if $F(x, y) \geq F_1(x)F_2(y)$ for all $x, y \in R$ (Lehmann 1966). Negative quadrant dependence is defined by replacing "$\geq$" with "$\leq$". Let

$$d_k(x, y) = F(x, y)^k - F_1(x)^k F_2(y)^k, \quad \forall x, y \in R,$$

where $k = \{1, 2, \ldots\}$. For each k, it is easy to show

$$X \text{ and } Y \text{ are positively quadrant dependent} \;\Leftrightarrow\; d_k(x, y) \geq 0,\ \forall x, y \in R,$$
$$X \text{ and } Y \text{ are negatively quadrant dependent} \;\Leftrightarrow\; d_k(x, y) \leq 0,\ \forall x, y \in R,$$
$$X \text{ and } Y \text{ are independent} \;\Leftrightarrow\; d_k(x, y) = 0,\ \forall x, y \in R.$$

Therefore, Kochar and Gupta (1987) introduced a dependence measure

$$D_k = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d_k(x, y)dF(x, y) = \int_0^1\int_0^1 \{C(u, v)^k - u^k v^k\}dC(u, v).$$

The last equation shows that Kochar and Gupta's dependence measure is free from the marginal distributions and depends only on the copula.

Theorem 2 (Kochar and Gupta's dependence measure) *Under the generalized FGM copula in Eq. (1),*

$$\begin{aligned} D_k &= k\left\{\frac{q}{k+1+pq}B\left(\frac{k+1}{p}, q\right)\right\}^2\theta \\ &\quad + \sum_{j=1}^{k-1}\binom{k}{j}(k-j)(k+1)\left\{\frac{q}{k+1+(j+1)pq}B\left(\frac{k+1}{p}, (j+1)q\right)\right\}^2\theta^{j+1}. \end{aligned}$$

Proof of Theorem 2 Amini et al. (2011) obtained

$$\begin{aligned} D_k &= \sum_{j=1}^{k}\binom{k}{j}\frac{\theta^j}{p^2}\left\{B\left(\frac{k+1}{p}, jq+1\right)\right\}^2 + \sum_{j=1}^{k}\binom{k}{j}\frac{\theta^{j+1}}{p^2} \\ &\quad \times\left\{B\left(\frac{k+1}{p}, (j+1)q\right) - (1+pq)B\left(\frac{k+1}{p}+1, (j+1)q\right)\right\}^2. \end{aligned}$$

By applying the properties of the beta function to the above function, we have

$$\begin{aligned} D_k &= \sum_{j=1}^{k}\binom{k}{j}\left\{\frac{jq}{k+1+jpq}B\left(\frac{k+1}{p}, jq\right)\right\}^2\theta^j \\ &\quad + \sum_{j=1}^{k}\binom{k}{j}\left\{\frac{(j-k)q}{k+1+(j+1)pq}B\left(\frac{k+1}{p}, (j+1)q\right)\right\}^2\theta^{j+1}. \end{aligned}$$

With some further simplifications, we obtain the desired result. The detailed proof is available in Supplementary Material. □

Our new expression of D_k in Theorem 2 is useful to study the properties of D_k. Our expression shows that D_k is a polynomial of order k in the form

$$D_k = d_1\theta + d_2\theta^2 + \cdots + d_k\theta^k,$$

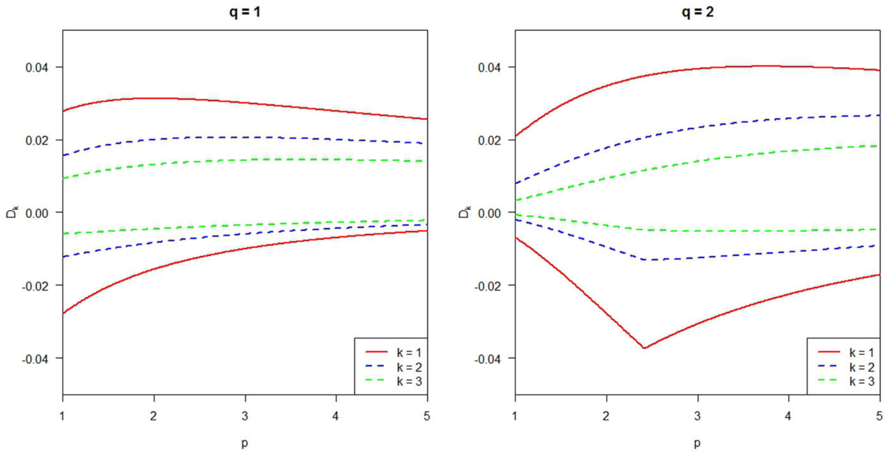

Fig. 3 The range of D_k, $k = 1, 2, 3$, under the generalized FGM copula with parameters $p \in [1, 5]$ and $q = 1$ or 2

where the coefficients $d_1, d_2, \ldots, d_k$ are explicitly defined in Theorem 2. For instance,

$$D_1 = \left\{ \frac{q}{2+pq} B\left(\frac{2}{p}, q\right) \right\}^2 \theta,$$

$$D_2 = 2\left\{ \frac{q}{3+pq} B\left(\frac{3}{p}, q\right) \right\}^2 \theta + 6\left\{ \frac{q}{3+2pq} B\left(\frac{3}{p}, 2q\right) \right\}^2 \theta^2,$$

$$D_3 = 3\left\{ \frac{q}{4+pq} B\left(\frac{4}{p}, q\right) \right\}^2 \theta + 24\left\{ \frac{q}{4+2pq} B\left(\frac{4}{p}, 2q\right) \right\}^2 \theta^2 + 12\left\{ \frac{q}{4+3pq} B\left(\frac{4}{p}, 3q\right) \right\}^2 \theta^3.$$

Thus, we have $D_1 = \rho/12 = \tau/8$. The expressions of D_1, D_2 and D_3 reveal that $D_1(-\theta) = -D_1(\theta)$ but $D_2(-\theta) \neq -D_2(\theta)$ and $D_3(-\theta) \neq -D_3(\theta)$. Figure 3 illustrates the ranges of D_1, D_2 and D_3.

4 Competing risks measures

In this section, we derive some useful measures for analyzing dependent competing risks models under the generalized FGM copula. We focus on measures related to "sub-distribution function" which plays a fundamental role in competing risks models (Gray 1988; Crowder 2001; Escarela and Carrière 2003; Bakoyannis and Touloumi 2012).

Let X and Y be continuous failure times following the generalized FGM copula in Eq. (1). Under competing risks models, the failure times X and Y are called "latent failure times" (Chap. 3 of Crowder 2001); what we actually observe is the first occurring failure time $T = \min(X, Y)$ and the failure cause $C = 1$ if $X \leq Y$ or $C = 2$ if $Y < X$.

Traditionally, specific bivariate distributions have been employed for modeling the joint distribution of (X, Y), such as bivariate normal distribution (Basu and Ghosh 1978) and bivariate Weibull distribution (Moeschberger 1974). Instead, we follow copula-based approaches that specify a copula form with flexible marginal distributions (Zheng and Klein 1995).

The sub-distribution functions of the failure cause 1 ($C = 1$) and failure cause 2 ($C = 2$) can be written respectively as

$$F(1, t) = P(C = 1, T \leq t) = \int_0^t f(1, z)dz,$$

$$F(2, t) = P(C = 2, T \leq t) = \int_0^t f(2, z)dz,$$

where $f(1,t) = -\partial\overline{F}(x, y)/\partial x|_{x=y=t}$ and $f(2,t) = -\partial\overline{F}(x, y)/\partial y|_{x=y=t}$ are called "sub-density function", where

$$\begin{aligned}\overline{F}(x, y) &= 1 - F_1(x) - F_2(y) + F(x, y)\\ &= 1 - F_1(x) - F_2(y) + F_1(x)F_2(y)[1 + \theta\{1 - F_1(x)^p\}^q\{1 - F_2(y)^p\}^q]\end{aligned}$$

is the joint survival function under the generalized FGM copula.

Theorem 3 *Under the generalized FGM copula in Eq. (1), the sub-density functions can be expressed as*

$$\begin{aligned}f(1, t) = f_1(t) - F_2(t)f_1(t) - \theta \sum_{i=0}^{q}\sum_{j=0}^{q-1}\binom{q}{i}\binom{q-1}{j}(-1)^{i+j}\\ \times \{F_2(t)^{pi+1}F_1(t)^{pj}f_1(t) - (1 + pq)F_2(t)^{pi+1}F_1(t)^{pj+p}f_1(t)\},\\ f(2, t) = f_2(t) - F_1(t)f_2(t) - \theta \sum_{i=0}^{q}\sum_{j=0}^{q-1}\binom{q}{i}\binom{q-1}{j}(-1)^{i+j}\\ \times \{F_1(t)^{pi+1}F_2(t)^{pj}f_2(t) - (1 + pq)F_1(t)^{pi+1}F_2(t)^{pj+p}f_2(t)\}.\end{aligned}$$

Proof of Theorem 3 From the expression $f(2,t) = -\partial\overline{F}(x, y)/\partial y|_{x=y=t}$, we have

$$\begin{aligned}f(2, t) &= f_2(t) - F_1(t)f_2(t)\\ &\quad - \theta F_1(t)\{1 - F_1(t)^p\}^q f_2(t)\{1 - F_2(t)^p\}^{q-1}\{1 - (1 + pq)F_2(t)^p\}.\end{aligned}$$

Following a similar technique as Domma and Giordano (2013), we apply the binominal theorem to obtain the desired results. The form of $f(1, t)$ is obtained similarly. More detailed derivations are given in Supplementary Material. □

While the expressions in Theorem 3 look complicated, they allow one to view $f(1,t)$ (or $f(2,t)$) as a polynomial in $F_1(t)$ and $F_2(t)$. Accordingly, Theorem 3 immediately yields the following theorem:

Theorem 4 *Under the generalized FGM copula in Eq. (1), the sub-distribution functions can be expressed as*

$$F(1,t) = F_1(t) - \int_0^t F_2(z) f_1(z) dz - \theta \sum_{i=0}^{q} \sum_{j=0}^{q-1} \binom{q}{i} \binom{q-1}{j} (-1)^{i+j}$$
$$\times \left\{ \int_0^t F_2(z)^{pi+1} F_1(z)^{pj} f_1(z) dz - (1+pq) \int_0^t F_2(z)^{pi+1} F_1(z)^{pj+p} f_1(z) dz \right\},$$
$$F(2,t) = F_2(t) - \int_0^t F_1(z) f_2(z) dz - \theta \sum_{i=0}^{q} \sum_{j=0}^{q-1} \binom{q}{i} \binom{q-1}{j} (-1)^{i+j}$$
$$\times \left\{ \int_0^t F_1(z)^{pi+1} F_2(z)^{pj} f_2(z) dz - (1+pq) \int_0^t F_1(z)^{pi+1} F_2(z)^{pj+p} f_2(z) dz \right\}.$$

We examine the cases where the integrals in $F(1,t)$ and $F(2,t)$ have closed forms. To this end, we select the Burr III marginal distributions, defined as $F_1(x) = (1 + x^{-\delta})^{-\alpha}$ and $F_2(y) = (1 + y^{-\delta})^{-\beta}$, where α, β, δ are all positive shape parameters. Under the generalized FGM copula, the bivariate distribution function of (X, Y) is

$$F(x,y) = (1+x^{-\delta})^{-\alpha}(1+y^{-\delta})^{-\beta}[1+\theta\{1-(1+x^{-\delta})^{-\alpha p}\}^q\{1-(1+y^{-\delta})^{-\beta p}\}^q]. \tag{4}$$

The Burr III distribution belongs to the Burr system (Burr 1942) which is widely used in real applications. For instance, Lindsay et al. (1996) fitted the Burr III distribution for the diameter distribution of forest stands. Distributions from the Burr system have been discussed by various authors for lifetime data analysis with competing risks (Crowder 2001; Escarela and Carrière 2003; Lawless 2003). In addition, the reliability measure with the Burr III margins has been considered by Domma and Giordano (2013).

The following theorems are obtained by substituting the Burr III distribution to the formulas of Theorems 3 and 4.

Theorem 5 *Under the generalized FGM copula with the Burr III margins in Eq. (4), the sub-density functions can be expressed as*

$$f(1,t) = K_{\alpha\delta}(t) H_\delta(t)^{\alpha+1} - K_{\alpha\delta}(t) H_\delta(t)^{\alpha+\beta+1}$$
$$- \theta \sum_{i=0}^{q} \sum_{j=0}^{q-1} \binom{q}{i} \binom{q-1}{j} (-1)^{i+j}$$

$$\times \{K_{\alpha\delta}(t)H_\delta(t)^{\alpha(pj+1)+\beta(pi+1)+1}$$
$$-(1+pq)K_{\alpha\delta}(t)H_\delta(t)^{\alpha(pj+p+1)+\beta(pi+1)+1}\},$$
$$f(2,t) = K_{\beta\delta}(t)H_\delta(t)^{\beta+1} - K_{\beta\delta}(t)H_\delta(t)^{\alpha+\beta+1}$$
$$-\theta\sum_{i=0}^{q}\sum_{j=0}^{q-1}\binom{q}{i}\binom{q-1}{j}(-1)^{i+j}$$
$$\times \{K_{\beta\delta}(t)H_\delta(t)^{\alpha(pi+1)+\beta(pj+1)+1}$$
$$-(1+pq)K_{\beta\delta}(t)H_\delta(t)^{\alpha(pi+1)+\beta(pj+p+1)+1}\},$$

where $K_{\alpha\delta}(t) = \alpha\delta t^{-\delta-1}$, $K_{\beta\delta}(t) = \beta\delta t^{-\delta-1}$ *and* $H_\delta(t) = (1+t^{-\delta})^{-1}$.

Theorem 6 *Under the generalized FGM copula with the Burr III margins in Eq. (4), the sub-distribution functions can be expressed as*

$$F(1,t) = H_\delta(t)^\alpha - \frac{\alpha}{\alpha+\beta}H_\delta(t)^{\alpha+\beta} - \theta\alpha\sum_{i=0}^{q}\sum_{j=0}^{q-1}\binom{q}{i}\binom{q-1}{j}(-1)^{i+j}$$
$$\times\left\{\frac{H_\delta(t)^{\alpha(pj+1)+\beta(pi+1)}}{\alpha(pj+1)+\beta(pi+1)} - \frac{(1+pq)H_\delta(t)^{\alpha(pj+p+1)+\beta(pi+1)}}{\alpha(pj+p+1)+\beta(pi+1)}\right\},$$
$$F(2,t) = H_\delta(t)^\beta - \frac{\beta}{\alpha+\beta}H_\delta(t)^{\alpha+\beta} - \theta\beta\sum_{i=0}^{q}\sum_{j=0}^{q-1}\binom{q}{i}\binom{q-1}{j}(-1)^{i+j}$$
$$\times\left\{\frac{H_\delta(t)^{\alpha(pi+1)+\beta(pj+1)}}{\alpha(pi+1)+\beta(pj+1)} - \frac{(1+pq)H_\delta(t)^{\alpha(pi+1)+\beta(pj+p+1)}}{\alpha(pi+1)+\beta(pj+p+1)}\right\},$$

where $H_\delta(t) = (1+t^{-\delta})^{-1}$.

The proof of Theorem 6 is given in Supplementary Material.

If we consider the Burr III margins with different shape parameters, say δ_1 and δ_2, we have $F_1(x) = (1+x^{-\delta_1})^{-\alpha}$ and $F_2(y) = (1+y^{-\delta_2})^{-\beta}$. To obtain the sub-distribution functions, one needs to compute

$$\int_0^t (1+z^{-\delta_1})^{-\alpha}\beta\delta_2(1+z^{-\delta_2})^{-\beta-1}z^{-\delta_2-1}dz.$$

The preceding expression has a closed form only when $\delta_1 = \delta_2 = \delta$ by a transformation $1+z^{-\delta} = k$ (Supplementary Material). Thus, simple formulas for sub-distribution functions are unavailable when $\delta_1 \neq \delta_2$.

The case of $p = q = 1$ becomes the sub-distribution functions of the FGM copula,

$$F(1,t) = H_\delta(t)^\alpha - \frac{\alpha}{\alpha+\beta} H_\delta(t)^{\alpha+\beta} - \theta\alpha \left\{ \frac{1}{\alpha+\beta} H_\delta(t)^{\alpha+\beta} - \frac{1}{\alpha+2\beta} H_\delta(t)^{\alpha+2\beta} - \frac{2}{2\alpha+\beta} H_\delta(t)^{2\alpha+\beta} + \frac{1}{\alpha+\beta} H_\delta(t)^{2\alpha+2\beta} \right\},$$

$$F(2,t) = H_\delta(t)^\beta - \frac{\beta}{\alpha+\beta} H_\delta(t)^{\alpha+\beta} - \theta\beta \left\{ \frac{1}{\alpha+\beta} H_\delta(t)^{\alpha+\beta} - \frac{1}{2\alpha+\beta} H_\delta(t)^{2\alpha+\beta} - \frac{2}{\alpha+2\beta} H_\delta(t)^{\alpha+2\beta} + \frac{1}{\alpha+\beta} H_\delta(t)^{2\alpha+2\beta} \right\}.$$

Letting $t \to \infty$ in the above expression, we have

$$F(2,\infty) = P(Y < X) = \frac{\alpha}{\alpha+\beta} + \theta \frac{\alpha\beta(\alpha-\beta)}{(\alpha+\beta)(2\alpha+\beta)(\alpha+2\beta)}.$$

This expression coincides with the reliability measure for the Burr III stress and strength variables under the FGM copula obtained by Domma and Giordano (2013). From these findings, our expressions $F(1,t)$ and $F(2,t)$ are interpreted as "truncated reliability measures", which generalize the expression of Domma and Giordano (2013).

5 Example from The Diabetic Retinopathy Study

We analyze a real data from the diabetic retinopathy study that is available in the R survival package (Therneau and Lumley 2016). The objective of the study is to access the effectiveness of the laser photocoagulation treatment for diabetic patients. For each patient, one eye was randomly selected to receive the treatment and the other eye was treated as a control. The data consist of time-to-blindness of the treatment eye (X) and time-to-blindness of the control eye (Y) for 197 diabetic patients. The patients were separated into two groups: Adult group (Age ≥20, 83 patients) and Juvenile group (Age <20, 114 patients). Both X and Y may be censored due to death, dropout, or the study end. In Adult group, 78% of X and 40% of Y were censored: in Juvenile group, 68% of X and 55% of Y were censored.

We assume that (X, Y) follows the generalized FGM copula with the Burr III margins in Eq. (4). We set $p = 3$ and $q = 2$ to yield a wide range $\tau \in [-0.245, 0.315]$, whereby the range of θ is $[-0.605, 0.778]$. For each group, the parameters $(\alpha, \beta, \delta, \theta)$ are estimated by the maximum likelihood estimator (MLE) based on bivariate censored data [Section 11.2.2.1 of Lawless (2003)].

The MLEs are summarized in Table 1. We obtained $\hat{\theta} = 0.621$ ($\hat{\tau} = 0.252$) in Adult group and $\hat{\theta} = 0.593$ ($\hat{\tau} = 0.240$) in Juvenile group. These results agree with Louzada et al. (2013) who applied a Bayesian estimate $\hat{\theta} = 0.644$ for both groups under the original FGM copula. Under the original FGM copula, however, the range of Kendall's tau is $\tau \in [-0.222, 0.222]$ that is too narrow. Hence, the generalized FGM copula may be more suitable for fitting the data.

Table 1 Parameter estimates for the generalized FGM copula model with the Burr III margins based on data from the diabetic retinopathy study

Group	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\theta}$
Adult	12.250	5.690	0.565	0.621 ($\hat{\tau} = 0.252$)
Juvenile	7.407	5.607	0.489	0.593 ($\hat{\tau} = 0.240$)

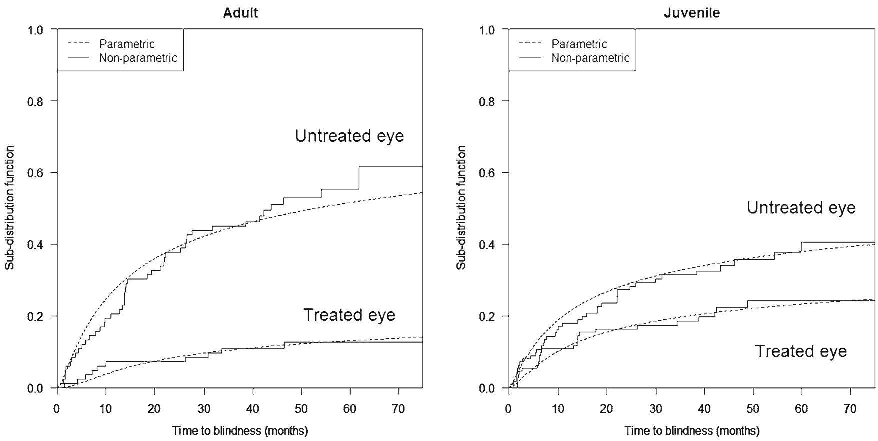

Fig. 4 Estimates of the sub-distribution functions using the parametric and non-parametric approaches based on the diabetic retinopathy data

Based on the MLEs obtained above, we estimated the sub-distribution functions using the formulas available in Theorem 6. We also applied the non-parametric estimator of sub-distribution functions [Section 9.2 of Lawless (2003)]. Figure 4 reveals that the two estimators are close to one another. This implies that the generalized FGM copula with Burr III margins fit well to the data.

6 Concluding remarks

New results obtained in this paper include the expression of Blest's coefficient (Sect. 3.2), simplified expression of Kochar and Gupta's dependence measure (Sect. 3.3), sub-density and sub-distribution functions under the generalized FGM copula (Sect. 4). With these expressions, the paper discusses mathematical relationships among the bivariate dependence measures, which are not explicitly discussed in the literature.

One should keep in mind that there still exists many other bivariate dependence measures with unknown expressions under the generalized FGM copula, such as Gini's coefficient (Gini 1912) and the Schweizer–Wolff measure (Schweizer and Wolff 1981). Gini's coefficient is a bivariate dependence measure based on the concept concordance whose formula may be difficult to derive under the generalized FGM copula. The Schweizer–Wolff measure is a L_p distance

$$\left(k_p \int_0^1 \int_0^1 |C(u, v) - uv|^p \, dudv \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

where k_p is a normalizing constant. It only takes positive values, so it is different from other bivariate dependence measures discussed in this paper. For $p = \infty$, it becomes $4 \sup_{0<u,v<1} |C(u, v) - uv|$. For integer values of p, the expressions may be available.

As for the bivariate competing risks models, we show that the sub-distribution function has an explicit form with the Burr III margins (Theorem 6). An interesting conclusion of this paper is that our expressions for the sub-distribution functions generalize the reliability measure previously considered by Domma and Giordano (2013).

We have conducted Monte Carlo simulations to verify the correctness of all the newly obtained expressions of dependence measures and competing risks measures under the generalized FGM copula. The simulation results are available in Supplementary Material.

Our results imply that the generalized FGM copula is appealing for many statistical problems, especially for analyzing competing risks data. In copula-based latent failure time models for competing risks data, statistical inference procedures often requires the form of a sub-density or a cause-specific hazard function; see real examples from biomedical studies (Escarela and Carrière 2003; Emura et al. 2015, 2017; Emura and Chen 2016), economic studies (Lo and Wilke 2010; De Uña-Álvarez and Veraverbeke 2014) and animal studies (Braekers and Veraverbeke 2005; Emura and Michimae 2017). In addition, the sub-distribution functions are used in the process of deriving estimators (Lo and Wilke 2010; De Uña-Álvarez and Veraverbeke 2013, 2014). Usually in the copula-based latent failure time models, the sub-distribution is not a target of estimation, but useful for model diagnostic procedures (Escarela and Carrière 2003). We are currently examining the applications of our theoretical results to the statistical problems with real competing risks.

Acknowledgements The authors thank the Editor and two anonymous reviewers for their helpful comments that improved the paper. The authors are also thankful to Prof. Christian Genest for his comments on the earlier version of the paper. This work is supported by the research grant funded by Ministry of Science and Technology, Taiwan (MOST 103-2118-M-008-MY2 and 105-2118-M-008-001-MY2).

References

Amblard C, Girard S (2009) A new extension of bivariate FGM copulas. Metrika 70(1):1–17

Amini M, Jabbari H, Mohtashami Borzadaran GR (2011) Aspects of dependence in generalized Farlie-Gumbel-Morgenstern distributions. Commun Stat Simul Comput 40(8):1192–1205

Bairamov I, Kotz S (2002) Dependence structure and symmetry of Huang-Kotz FGM distributions and their extensions. Metrika 56(1):55–72

Bakoyannis G, Touloumi G (2012) Practical methods for competing risks data: a review. Stat Methods Med Res 21(3):257–272

Basu AP, Ghosh JK (1978) Identifiability of the multinormal and other distributions under competing risks model. J Multivar Anal 8(3):413–429

Blest DC (2000) Rank correlation-an alternative measure. Aust N Z J Stat 42(1):101–111

Braekers R, Veraverbeke N (2005) A copula-graphic estimator for the conditional survival function under dependent censoring. Can J Stat 33(3):429–447
Burr IW (1942) Cumulative frequency functions. Ann Math. Stat 13(2):215–232
Capéraà P, Genest C (1993) Spearman's rho is larger than Kendall's tau for positively dependent random variables. J Nonparametric Stat 2(2):183–194
Clayton DG (1978) A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. Biometrika 65(1):141–151
Crowder MJ (2001) Classical competing risks. Chapman and Hall/CRC, Boca Raton
De Uña-Álvarez J, Veraverbeke N (2013) Generalized copula-graphic estimator. Test 22(2):343–360
De Uña-Álvarez J, Veraverbeke N (2014) Generalized copula-graphic estimator with left-truncated and right-censored data. Discuss Pap Stat Oper Res. http://jacobo.webs.uvigo.es/presentation_1.pdf
Domma F, Giordano S (2013) A copula-based approach to account for dependence in stress-strength models. Stat Pap 54(3):807–826
Domma F, Giordano S (2016) Concomitants of m-generalized order statistics from generalized Farlie-Gumbel-Morgenstern distribution family. J Comput Appl Math 294:413–435
Emura T, Chen YH (2016) Gene selection for survival data under dependent censoring: a copula-based approach. Stat Methods Med Res 25(6):2840–2857
Emura T, Michimae H (2017) A copula-based inference to piecewise exponential models under dependent censoring, with application to time to metamorphosis of salamander larvae. Environ Ecol Stat (in press)
Emura T, Nakatochi M, Murotani K, Rondeau V (2015) A joint frailty-copula model between tumor progression and death for meta-analysis. Stat Methods Med Res. doi:10.1177/0962280215604510
Emura T, Nakatochi M, Matsui S, Michimae H, Rondeau V (2017) Personalized dynamic prediction of death according to tumour progression and high-dimensional genetic factors: meta-analysis with a joint model. Stat Methods Med Res (in press)
Esary JD, Proschan F, Walkup DW (1967) Association of random variables, with applications. Ann Math Stat 38(5):1466–1474
Escarela G, Carrière JF (2003) Fitting competing risks with an assumed copula. Stat Methods Med Res 12(4):333–349
Eyraud H (1936) Les principes de la mesure des corrélations. Ann Univ Lyon III Ser Sect A 1:30–47
Farlie DJG (1960) The performance of some correlation coefficients for a general bivariate distribution. Biometrika 47(3–4):307–323
Frank MJ (1979) On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$. Aequ Math 19:194–226
Genest C (1987) Frank's family of bivariate distributions. Biometrika 74(3):549–555
Genest C, Plante JF (2003) On Blest's measure of rank correlation. Can J Stat 31(1):35–52
Gini C (1912) Variabilità e Mutuabilità. Contributo alloStudio delle Distribuzioni e delle Relazioni Statistiche. C, Cuppini, Bologna
Gray RJ (1988) A class of K-sample tests for comparing the cumulative incidence of a competing risk. Ann Stat 16(3):1141–1154
Gumbel EJ (1960a) Distributions des valeurs extrêmes en plusiers dimensions. Publications de l'Institut de Statistique de L'Université de Paris 9:171–173
Gumbel EJ (1960b) Bivariate exponential distributions. J Am Stat Assoc 55(292):698–707
Joe H (1993) Parametric families of multivariate distributions with given margins. J Multiv Anal 46(2):262–282
Kochar SC, Gupta RP (1987) Competitors of Kendall-tau test for testing independence against positive quadrant dependence. Biometrika 74(3):664–666
Lawless JF (2003) Statistical models and methods for lifetime data, 2nd edn. Wiley, Hoboken
Lehmann EL (1966) Some concepts of dependence. Ann Math Stat 37(5):1137–1153
Lindsay SR, Wood GR, Woollons RC (1996) Modelling the diameter distribution of forest stands using the Burr distribution. J Appl Stat 23(6):609–620
Louzada F, Suzuki AK, Cancho VG (2013) The FGM long-term bivariate survival copula model: modeling, Bayesian estimation, and case influence diagnostics. Commun Stat Theory Methods 42(4):673–691
Lo SMS, Wilke RA (2010) A copula model for dependent competing risks. J R Stat Soc Ser C Appl Stat 59(2):359–376
Moeschberger ML (1974) Life tests under dependent competing causes of failure. Technometrics 16(1):39–47
Morgenstern D (1956) Einfache Beispiele zweidimensionaler Verteilungen. Mitt Math Stat 8:234–235

Nelsen RB (2006) An introduction to copulas, 2nd edn. Springer, New York
Nešlehová J (2007) On rank correlation measures for non-continuous random variables. J Multivar Anal 98(3):544–567
Rodíguez-Lallena JA, Úbeda-Flores M (2004) A new class of bivariate copulas. Stat Probab Lett 66(3):315–325
Scarsini M (1984) On measures of concordance. Stochastica 8(3):201–218
Schucany WR, Parr WC, Boyer JE (1978) Correlation structure in Farlie-Gumbel-Morgenstern distributions. Biometrika 65(3):650–653
Schweizer B, Wolff EF (1981) On nonparametric measures of dependence for random variables. Ann Stat 9(4):879–885
Sklar A (1959) Fonctions de répartition à n dimensions et leurs marges. Publications de l'Institut de Statistique de L'Université de Paris 8:229–231
Therneau T, Lumley T (2016) R survival package, version 2.40-1. Date 2016-10-30
Zheng M, Klein JP (1995) Estimates of marginal survival for dependent competing risks based on an assumed copula. Biometrika 82(1):127–138