A copula-based parametric maximum likelihood estimation for dependently left-truncated data

Presenter: Chi-Hung, Pan（ 潘奇鴻 ）
Advisor: Takeshi Emura（ 江村剛志 ）博士

Graduate Institute of Statistics, National Central University

Outlines

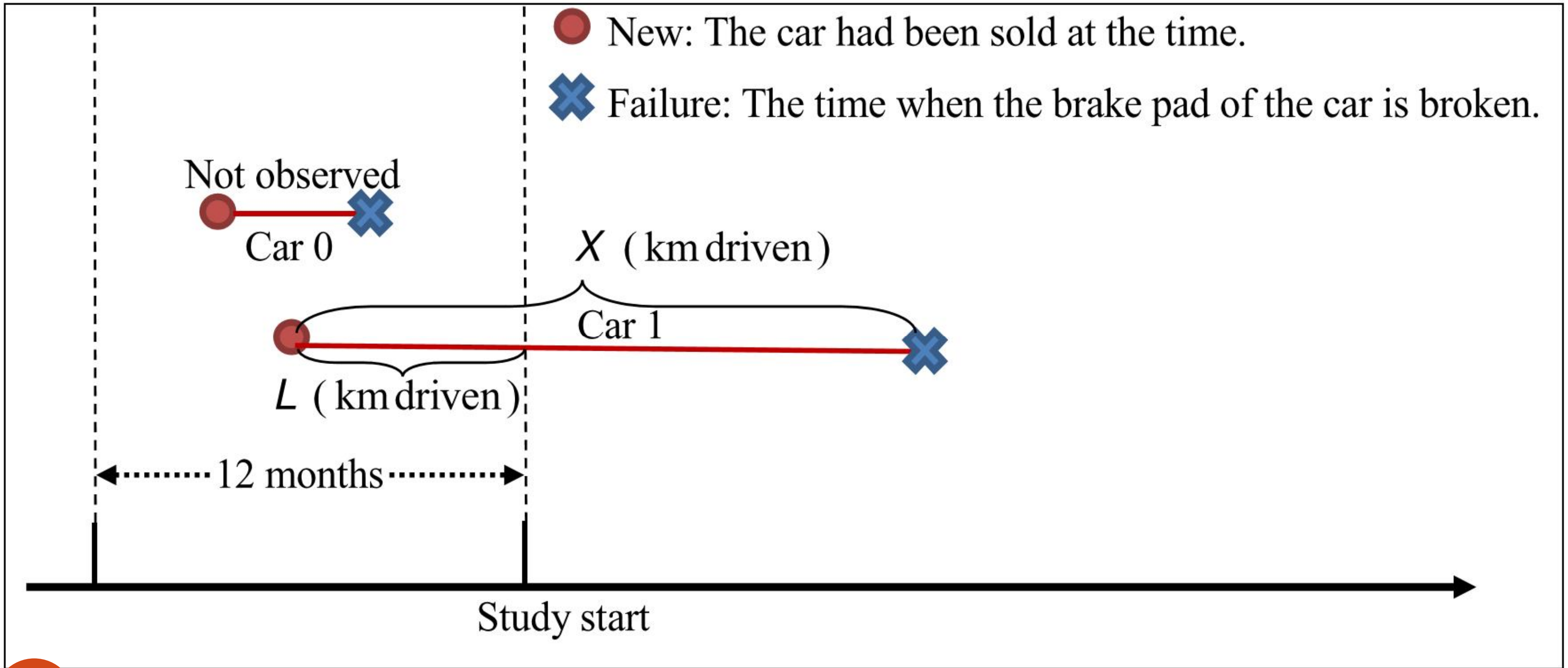
- Review
  - Left-truncated data
  - Copula
- Method
- Simulation
- Data analysis
- Summary
- Future work

Left-truncated

- X : lifetime (Product or Biological)
  (interest variable)
- Data: X is observed only when $L \leq X$.
- X is left-truncated by L.

Brake pads data (Kalbfleisch and Lawless,1992)

- X : The number of kilometers of the car driven at failure
- L : The number of kilometers of the car driven at the study start



Independence assumption

- Lynden-Bell estimator (1971)
- Kalbfleisch and Lawless (1992)

Independence assumption ?

- Check by test
  (Emura and Wang 2010)

- Estimate of the dependence parameter
  (Our thesis)

Model construction

- Parametric model
  - Bivariate normal distribution

    (Emura and Konno 2012)
- Semi-parametric model
  - Copula

    (Chaieb et al. 2006), (Emura et al., 2011), (Emura and Wang 2012), (Emura and Murotani 2015)
- Nonparametric model
  - Copula

    (Strzalkowska-Kominiak and stute 2013)

Copula

- A bivariate copula is a distribution function

$$C_\alpha : [\,0,1\,]^2 \mapsto [\,0,1\,],$$

where α is a unknown parameter.

- For a bivariate distribution F with marginal distribution F_L and F_X, we know that exist a copula C_α:

$$F(\,l,x\,) = C_\alpha[\,F_L(\,l\,), F_X(\,x\,)\,]$$

Sklar's theorem(Sklar 1959).

Kendall's tau

- For better comparison, the parameter transformed to kendall's tau.

- $\tau = 4\int C_{\alpha}(u_1, u_2)du_1du_2 - 1$

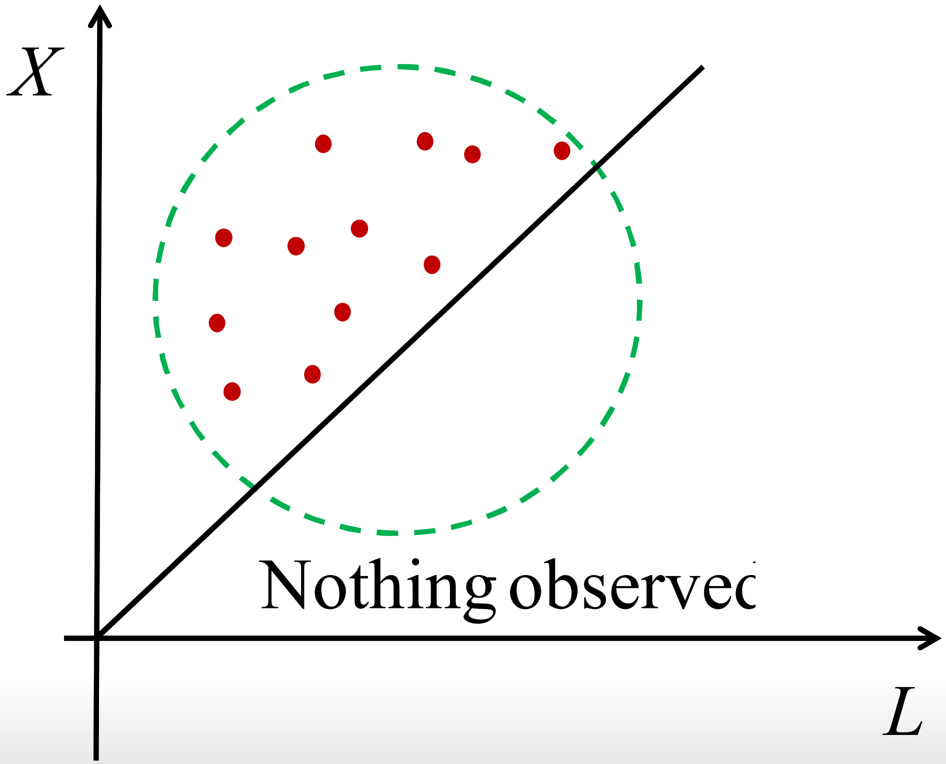
  - Ex1. Under the Clayton copula
    - $\tau = \alpha/(\alpha + 2)$
  - Ex2. Under the joe copula
    - $\tau = 1 - 4\int_0^{\infty} \frac{t}{\alpha^2}\{1 - \exp(-t)\}^{2/\alpha - 2} \exp(-2t)dt$

Propose method

- Model: $C_\alpha[\,F_L(\,l;\boldsymbol{\theta}_L\,), F_X(x;\boldsymbol{\theta}_X\,)\,]$

Propose method

- Model: $C_\alpha[\,F_L(\,l;\boldsymbol{\theta}_L\,),F_X(x;\boldsymbol{\theta}_X\,)\,]$
- Data: Left-truncated data



Propose method

- Model: $C_\alpha[\, F_L(\, l; \boldsymbol{\theta}_L\,), F_X(x; \boldsymbol{\theta}_X\,)\,]$

- Data: Left-truncated data

- Target: Derive MLE of parameters

- Method: “Randomize” Newton-Raphson algorithm (Hu and Emura 2015)

Notation

- $C_\alpha(u_1,u_2)$: copula function with dependence parameter α.

- $$C_\alpha^{[i,j]}(u_1,u_2) = \frac{\partial^{(i+j)} C_\alpha(u_1,u_2)}{\partial u_1^i \partial u_2^j},$$

- $$h(u_1,u_2) = \Pr(U_1 \le u_1 \mid U_2 = u_2) = \frac{\partial C_\alpha(u_1,u_2)}{\partial u_2}$$

(Schepsmeier and Stöber 2012)

Copula model (parametric)

- Consider the bivariate copula model defined as

$$\Pr_{\boldsymbol{\theta}}(L \le l, X \le x) = C_\alpha[F_L(l; \boldsymbol{\theta}_L), F_X(x; \boldsymbol{\theta}_X)],$$

where $\boldsymbol{\theta} = (\alpha, \boldsymbol{\theta}_L, \boldsymbol{\theta}_X)$ is a vector of parameters

- Density function:

$$f_{L,X}(l, x; \boldsymbol{\theta}) = C_\alpha^{[1,1]}[F_L(l; \boldsymbol{\theta}_L), F_X(x; \boldsymbol{\theta}_X)] f_L(l; \boldsymbol{\theta}_L) f_X(x; \boldsymbol{\theta}_X),$$

where $f_L(l; \boldsymbol{\theta}_L)$ and $f_X(x; \boldsymbol{\theta}_X)$ are the pdf of L and X.

Likelihood construction

- Given the observed data $\{(L_j, X_j); j = 1, 2, \ldots, n\}$, subject to $L_j \le X_j$ the likelihood function has the form:

$$L_n(\boldsymbol{\theta}) = c(\boldsymbol{\theta})^{-n} \prod_j f_{L,X}(L_j, X_j; \boldsymbol{\theta}),$$

where

$$c(\boldsymbol{\theta}) = \Pr(L \le X) = \iint_{l \le x} f_{L,X}(l, x; \boldsymbol{\theta}) dxdl$$

(inclusion probability).

The log-likelihood:

The log-likelihood function:

$$\ell_n(\boldsymbol{\theta}) = \log L_n(\boldsymbol{\theta}) = -n \log c(\boldsymbol{\theta}) + \sum_j \log f_{L,X}(L_j, X_j; \boldsymbol{\theta})$$

Theory

Theorem 1

The inclusion probability can be simplify as follows:

$$c(\boldsymbol{\theta}) = \Pr(L \leq X) = \int_0^1 H(u;\boldsymbol{\theta})du,$$

where

$$H(u;\boldsymbol{\theta}) \equiv h_\alpha[F_L\{F_X^{-1}(u;\boldsymbol{\theta}_X);\boldsymbol{\theta}_L\}, u].$$

Theory

Lemma 1 [p.301 Khuri (2003)]

Let a_i *and* b_i *be real numbers with* $a_i < b_i, i = 1, 2, \ldots, p$ *. Let*
$H : D \to R$ *, where* $D = \{ (u, \theta_1, \theta_2, \ldots, \theta_p) \mid 0 \le u \le 1, a_i \le \theta_i \le b_i, \text{for } \forall i = 1, \ldots, p \}$
For fixed $\theta_j, j \ne i$*, let* $D_i = \{ (u, \theta_i) \mid 0 \le u \le 1, a_i \le \theta_i \le b_i, a_i, b_i \in R \}$.

If H *and* $\partial H / \partial \theta_i$ *are continuous in* D_i,
then

$$\frac{\partial}{\partial \theta_i} \int_0^1 H(u; \boldsymbol{\theta})\, du = \int_0^1 \frac{\partial H(u; \boldsymbol{\theta})}{\partial \theta_i} du, \; i = 1, \ldots, p.$$

The log-likelihood:

The log-likelihood function:

$$\ell_n(\boldsymbol{\theta}) = \log L_n(\boldsymbol{\theta}) = -n\log c(\boldsymbol{\theta}) + \sum_j \log f_{L,X}(L_j, X_j; \boldsymbol{\theta})$$

$$= -n\log\left\{\int_0^1 H(u;\boldsymbol{\theta})du\right\}$$

$$+\sum_j \log f_L(L_j;\boldsymbol{\theta}_L) + \sum_j \log f_X(X_j;\boldsymbol{\theta}_X)$$

$$+\sum_j \log C_\alpha^{[1,1]}[F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)].$$

Score vector

- $$\frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \alpha} = -\frac{n}{c(\boldsymbol{\theta})}\frac{\partial c(\boldsymbol{\theta})}{\partial \alpha} + \sum_j \left[\frac{1}{C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}} \times \frac{\partial C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}{\partial \alpha} \right],$$
- $$\frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_L} = -\frac{n}{c(\boldsymbol{\theta})}\frac{\partial c(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_L} + \sum_j \frac{1}{f_L(L_j;\boldsymbol{\theta}_L)}\frac{\partial f_L(L_j;\boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L} + \sum_j \left[\frac{C_\alpha^{[2,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}{C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}} \frac{\partial F_L(L_j;\boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L} \right],$$

Score vector

- $$\frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_X} = -\frac{n}{c(\boldsymbol{\theta})}\frac{\partial c(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_X} + \sum_j \frac{1}{f_X(X_j;\boldsymbol{\theta}_X)}\frac{\partial f_X(X_j;\boldsymbol{\theta}_X)}{\partial \boldsymbol{\theta}_X} + \sum_j \left[\frac{C_\alpha^{[1,2]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}{C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}} \frac{\partial F_X(X_j;\boldsymbol{\theta}_X)}{\boldsymbol{\theta}_X} \right],$$

Hessian matrix

- $$\frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \alpha^2} = -n\left[\frac{1}{c(\boldsymbol{\theta})}\frac{\partial^2 c(\boldsymbol{\theta})}{\partial \alpha^2} - \frac{1}{c(\boldsymbol{\theta})^2}\left\{\frac{\partial c(\boldsymbol{\theta})}{\partial \alpha}\right\}^2\right]$$
$$+\sum_j\left(\frac{1}{C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}\frac{\partial^2 C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}{\partial \alpha^2}\right.$$
$$\left.-\frac{1}{C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}^2}\left[\frac{\partial C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}{\partial \alpha}\right]^2\right)$$

Hessian matrix

- $$\frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_L \partial \boldsymbol{\theta}_L^{\mathrm{T}}} = -n\left[\frac{1}{c(\boldsymbol{\theta})} \frac{\partial^2 c(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_L \partial \boldsymbol{\theta}_L^{\mathrm{T}}} - \frac{1}{c(\boldsymbol{\theta})^2} \frac{\partial c(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_L} \frac{\partial c(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_L^{\mathrm{T}}} \right]$$
$$+ \sum_j \left[\frac{1}{f_L(L_j;\boldsymbol{\theta}_L)} \frac{\partial^2 f_L(L_j;\boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L \partial \boldsymbol{\theta}_L^{\mathrm{T}}} - \frac{1}{f_L(L_j;\boldsymbol{\theta}_L)^2} \frac{\partial f_L(L_j;\boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L} \frac{\partial f_L(L_j;\boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L^{\mathrm{T}}} \right]$$
$$+ \sum_j \left[\frac{C_\alpha^{[3,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}{C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}} \frac{\partial F_L(L_j;\boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L} \frac{\partial F_L(L_j;\boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L^{\mathrm{T}}} \right.$$
$$- \frac{C_\alpha^{[2,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}^2}{C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}^2} \frac{\partial F_L(L_j;\boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L} \frac{\partial F_L(L_j;\boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L^{\mathrm{T}}}$$
$$\left. + \frac{C_\alpha^{[2,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}{C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}} \frac{\partial^2 F_L(L_j;\boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L \partial \boldsymbol{\theta}_L^{\mathrm{T}}} \right]$$

Hessian matrix

- $$\frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial\alpha\partial\boldsymbol{\theta}_L^{\mathrm{T}}} = -n\left\{ \frac{1}{c(\boldsymbol{\theta})}\frac{\partial^2 c(\boldsymbol{\theta})}{\partial\alpha\partial\boldsymbol{\theta}_L^{\mathrm{T}}} - \frac{1}{c(\boldsymbol{\theta})^2}\frac{\partial c(\boldsymbol{\theta})}{\partial\alpha}\frac{\partial c(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}_L^{\mathrm{T}}} \right\}$$

$$+\sum_j \frac{\partial F_L(L_j;\boldsymbol{\theta}_L)}{\partial\boldsymbol{\theta}_L^{\mathrm{T}}}\left[\frac{1}{C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}} \times \frac{\partial C_\alpha^{[2,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}{\partial\alpha} - \frac{C_\alpha^{[2,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}{C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}^2} \times \frac{\partial C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}{\partial\alpha} \right]$$

Hessian matrix

- $$\frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_L \partial \boldsymbol{\theta}_X^{\mathrm{T}}} = \sum_j \frac{\partial F_L(L_j;\boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L} \frac{\partial F_X(X_j;\boldsymbol{\theta}_X)}{\partial \boldsymbol{\theta}_X^{\mathrm{T}}} \left[\frac{C_\alpha^{[2,2]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}{C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}} \right.$$
$$\left. - \frac{C_\alpha^{[2,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\} C_\alpha^{[1,2]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}}{C_\alpha^{[1,1]}\{F_L(L_j;\boldsymbol{\theta}_L), F_X(X_j;\boldsymbol{\theta}_X)\}^2} \right]$$
$$- n \left\{ \frac{1}{c(\boldsymbol{\theta})} \frac{\partial^2 c(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_L \partial \boldsymbol{\theta}_X^{\mathrm{T}}} - \frac{1}{c(\boldsymbol{\theta})^2} \frac{\partial c(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_L} \frac{\partial c(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_X^{\mathrm{T}}} \right\},$$

Components of the Score and the Hessian

- $\dfrac{\partial c(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$ and $\dfrac{\partial^2 c(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^{\mathrm{T}}}$.

- $C_{\alpha}^{[i,j]}$ with $(i, j) = (1, 1), (1, 2), (2, 1), (2, 2), (1, 3), (3, 1)$.

- $\dfrac{\partial C_{\alpha}^{[i,j]}}{\partial \alpha}$ with $(i, j) = (1, 1), (1, 2), (2, 1)$ and $\dfrac{\partial^2 C_{\alpha}^{[1,1]}}{\partial \alpha^2}$.

- $\dfrac{\partial F_L(L_j; \boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L}$, $\dfrac{\partial^2 F_L(L_j; \boldsymbol{\theta}_L)}{\partial \boldsymbol{\theta}_L \partial \boldsymbol{\theta}_L^{\mathrm{T}}}$, $\dfrac{\partial F_X(X_j; \boldsymbol{\theta}_X)}{\partial \boldsymbol{\theta}_X}$

  and $\dfrac{\partial^2 F_X(X_j; \boldsymbol{\theta}_X)}{\partial \boldsymbol{\theta}_X \partial \boldsymbol{\theta}_X^{\mathrm{T}}}$.

Parametric model (Specified)

Consider the Clayton copula

$$C_\alpha(u_1, u_2) = (u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-1/\alpha}, \qquad \alpha \geq 0.$$

Then

$$h_\alpha(u_1, u_2) = u_2^{-\alpha-1}(u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-1/\alpha-1}.$$

Parametric model (Specified)

Assume $F_L(l;\lambda_L,\nu_L)=1-\exp(-\lambda_L l^{\nu_L})$ and

$$F_X(x;\lambda_X,\nu_X)=1-\exp(-\lambda_X x^{\nu_X})$$

where $\lambda_L>0$, $\lambda_X>0, \nu_L>0$ and $\nu_X>0$. (Weibull lifetime)

Then

$$F_X^{-1}(u)=\{-\lambda_X^{-1}\log(1-u)\}^{1/\nu_L},$$

$$F_L\{F_X^{-1}(u)\}=1-\exp[-\lambda_L\{-\lambda_X^{-1}\log(1-u)\}^{\nu_L/\nu_X}].$$

Parametric model (Specified)

By Theorem 1, the inclusion probability is

$$c(\boldsymbol{\theta}) = \Pr(L \le X) = \int_0^1 H(u;\boldsymbol{\theta})du$$

where $\boldsymbol{\theta} = (\alpha, \lambda_L, \lambda_X, \nu_L, \nu_X)$.

$$H(u;\boldsymbol{\theta}) = u^{-\alpha-1} B(u,\boldsymbol{\theta})^{-1/\alpha-1},$$

where

$$B(u;\boldsymbol{\theta}) = (1 - \exp[-\lambda_L \{-\log(1-u)/\lambda_X\}^{\nu_L/\nu_X}])^{-\alpha} + u^{-\alpha} - 1.$$

- Calculate score vector and hessian matrix (Lemma 1).
  Find MLE ("Randomize" Newton-Raphson algorithm).

Randomized Newton-Raphson for Weibull

- **Step1.** Let initial parameters be $\alpha^{(0)} = 2\hat{\tau}/(1-\hat{\tau})$, $\lambda_L^{(0)} = 1/\overline{L}$, $\lambda_X^{(0)} = 1/\overline{X}, \nu_L^{(0)} = 1$ and $\nu_X^{(0)} = 1$.

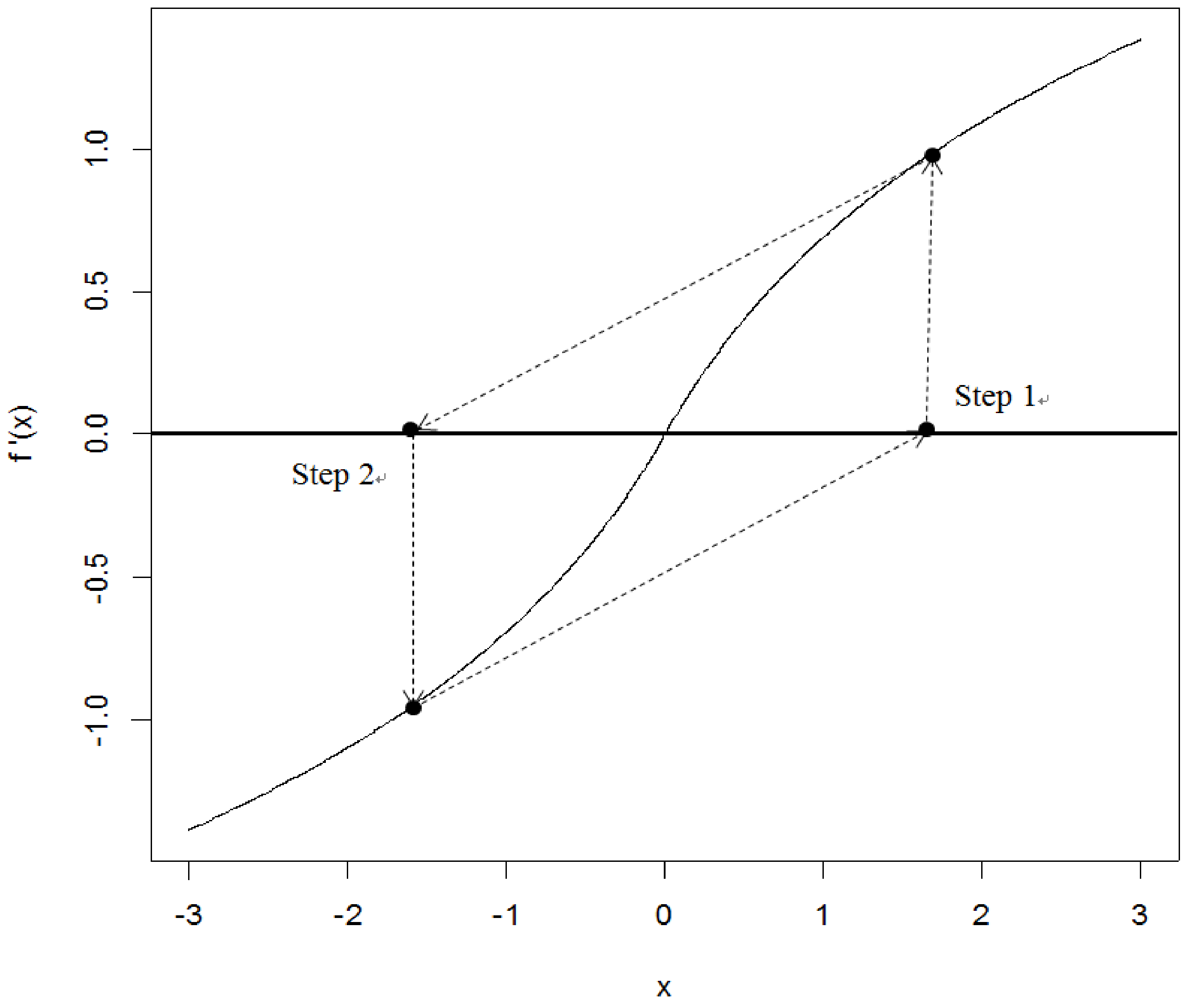
- **Step2.** Repeat the following iteration, for $k = 0, 1, 2...$:

$$\begin{bmatrix} \alpha^{(k+1)} \\ \lambda_L^{(k+1)} \\ \lambda_X^{(k+1)} \\ \nu_L^{(k+1)} \\ \nu_X^{(k+1)} \end{bmatrix} = \begin{bmatrix} \alpha^{(k)} \\ \lambda_L^{(k)} \\ \lambda_X^{(k)} \\ \nu_L^{(k)} \\ \nu_X^{(k)} \end{bmatrix} - \left. \begin{bmatrix} \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \alpha^2} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \lambda_L \partial \alpha} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \lambda_X \partial \alpha} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_L \partial \alpha} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_X \partial \alpha} \\ \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \lambda_L \partial \alpha} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \lambda_L^2} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \lambda_L \partial \lambda_X} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \lambda_L \partial \nu_L} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \lambda_L \partial \nu_X} \\ \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \lambda_X \partial \alpha} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \lambda_L \partial \lambda_X} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \lambda_X^2} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_L \partial \lambda_X} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_X \partial \lambda_X} \\ \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_L \partial \alpha} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_L \partial \lambda_L} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_L \partial \lambda_X} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_L^2} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_L \partial \nu_X} \\ \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_X \partial \alpha} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_X \partial \lambda_L} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_X \partial \lambda_X} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_L \partial \nu_X} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \nu_X^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \alpha} \\ \frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \lambda_L} \\ \frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \lambda_X} \\ \frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \nu_L} \\ \frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \nu_X} \end{bmatrix} \right|_{\substack{\alpha = \alpha^{(k)} \\ \lambda_L = \lambda_L^{(k)} \\ \lambda_X = \lambda_X^{(k)} \\ \nu_L = \nu_L^{(k)} \\ \nu_X = \nu_X^{(k)}}}$$

- If $\max\{|\alpha^{(k+1)}-\alpha^{(k)}|, |\lambda_L^{(k+1)}-\lambda_L^{(k)}|, |\lambda_X^{(k+1)}-\lambda_X^{(k)}|, |\nu_L^{(k+1)}-\nu_L^{(k)}|, |\nu_X^{(k+1)}-\nu_X^{(k)}|\}<\varepsilon$ and hessian matrix is negative define, then stop.

- If $\max\{|\alpha^{(k+1)}-\alpha^{(k)}|, |\lambda_L^{(k+1)}-\lambda_L^{(k)}|, |\lambda_X^{(k+1)}-\lambda_X^{(k)}|, |\nu_L^{(k+1)}-\nu_L^{(k)}|, |\nu_X^{(k+1)}-\nu_X^{(k)}|\}>2$ or $\alpha^{(k+1)}>20$ or $\alpha^{(k+1)}<10^{-6}$ or $\min\{\lambda_L^{(k+1)}, \lambda_X^{(k+1)}, \nu_L^{(k+1)}, \nu_X^{(k+1)}\}<10^{-8}$ or $k=\{100, 200, 300...\}$,

➢ Replace $(\alpha^{(0)}, \lambda_L{}^{(0)}, \lambda_X{}^{(0)}, \nu_L{}^{(0)}, \nu_X{}^{(0)})$ with $\{\alpha^{(0)}\times\exp(u_1), \lambda_L^{(0)}\times\exp(u_2), \lambda_X^{(0)}\times\exp(u_3), \nu_L^{(0)}\times\exp(u_4), \nu_X^{(0)}\times\exp(u_5)\}$ where $u_1\sim U(-1,1)$ and $u_2, u_3, u_4, u_5\sim U(-0.5, 0.5)$. Then, return to step 2.



Step 1
Step 2
f'(x)
x
1.0
0.5
0.0
-0.5
-1.0
-3
-2
-1
0
1
2
3

Simulation

- Set four different levels of the dependence parameter α.
- Then set three different levels of inclusion probability $c(\boldsymbol{\theta})$.
- Generates data $\{(L_j, X_j);\ j = 1, 2, \ldots, n\}$ subject to $L_j \le X_j$ (The Clayton copula with Weibull margins)
- Obtain the MLE of parameters and coverage probability.

Notation

- R: repetition times
- $E(\hat{\theta}_j) \equiv \frac{1}{R}\sum_{r=1}^{R}\hat{\theta}_{j(r)}$
- $MSE(\hat{\theta}_j) = \frac{1}{R}\sum_{r=1}^{R}\{\hat{\theta}_{j(r)} - \theta\}^2$

Table 2(a) Simulation results under the Clayton copula with the Weibull margins based on 1000 repetitions. $\boldsymbol{\theta}=(\alpha,\lambda_L,\lambda_X,\nu_L,\nu_X)$.

		$c(\boldsymbol{\theta})$	n	$E(\hat{\alpha})$	$E(\hat{\lambda}_L)$	$E(\hat{\lambda}_X)$	$E(\hat{\nu}_L)$	$E(\hat{\nu}_X)$	AI
$\alpha=0.5$ $(\tau=0.2)$	$\lambda_L=2,\lambda_X=1,$ $\nu_L=1,\nu_X=1$	0.700	100	0.519	2.034	1.031	1.015	1.011	7.4
			200	0.500	2.011	1.019	1.009	1.005	6.9
			300	0.499	2.005	1.016	1.005	1.002	6.7
	$\lambda_L=1,\lambda_X=1,$ $\nu_L=1,\nu_X=1$	0.500	100	Un-convergence					
			200	0.526	1.016	1.013	1.008	1.013	8.2
			300	0.518	1.010	1.009	1.005	1.011	8.1
	$\lambda_L=1,\lambda_X=1,$ $\nu_L=2,\nu_X=1$	0.436	100	0.550	1.007	1.040	2.058	1.044	22.7
			200	0.525	1.001	1.013	2.032	1.029	20.8
			300	0.515	0.998	1.011	2.021	1.020	21.1
$\alpha=1$ $(\tau=0.33)$	$\lambda_L=2,\lambda_X=1,$ $\nu_L=1,\nu_X=1$	0.736	100	1.042	2.035	1.031	1.010	1.005	6.6
			200	1.018	2.017	1.017	1.006	1.003	6.4
			300	1.012	2.009	1.014	1.002	1.001	6.4
	$\lambda_L=1,\lambda_X=1,$ $\nu_L=1,\nu_X=1$	0.500	100	1.079	1.012	1.048	1.013	1.011	9.0
			200	1.046	1.014	1.030	1.003	1.001	7.8
			300	1.030	1.009	1.023	1.002	1.001	7.8
	$\lambda_L=1,\lambda_X=1,$ $\nu_L=2,\nu_X=1$	0.416	100	1.037	0.976	1.101	2.116	1.038	32.7
			200	1.012	0.983	1.045	2.057	1.019	30.5
			300	1.005	0.987	1.020	2.036	1.018	27.1

Table 2(a) Simulation results under the Clayton copula with the Weibull margins based on 1000 repetitions. $\boldsymbol{\theta}=(\alpha,\lambda_L,\lambda_X,\nu_L,\nu_X)$.

		$c(\boldsymbol{\theta})$	n	$E(\hat{\alpha})$	$E(\hat{\lambda}_L)$	$E(\hat{\lambda}_X)$	$E(\hat{\nu}_L)$	$E(\hat{\nu}_X)$	AI
$\alpha=0.5$ $(\tau=0.2)$	$\lambda_L=2,\lambda_X=1,$ $\nu_L=1,\nu_X=1$	0.700	100	0.519	2.034	1.031	1.015	1.011	7.4
			200	0.500	2.011	1.019	1.009	1.005	6.9
			300	0.499	2.005	1.016	1.005	1.002	6.7
	$\lambda_L=1,\lambda_X=1,$ $\nu_L=1,\nu_X=1$	0.500	100			Un-convergence			
			200	0.526	1.016	1.013	1.008	1.013	8.2
			300	0.518	1.010	1.009	1.005	1.011	8.1
	$\lambda_L=1,\lambda_X=1,$ $\nu_L=2,\nu_X=1$	0.436	100	0.550	1.007	1.040	2.058	1.044	22.7
			200	0.525	1.001	1.013	2.032	1.029	20.8
			300	0.515	0.998	1.011	2.021	1.020	21.1
$\alpha=1$ $(\tau=0.33)$	$\lambda_L=2,\lambda_X=1,$ $\nu_L=1,\nu_X=1$	0.736	100	1.042	2.035	1.031	1.010	1.005	6.6
			200	1.018	2.017	1.017	1.006	1.003	6.4
			300	1.012	2.009	1.014	1.002	1.001	6.4
	$\lambda_L=1,\lambda_X=1,$ $\nu_L=1,\nu_X=1$	0.500	100	1.079	1.012	1.048	1.013	1.011	9.0
			200	1.046	1.014	1.030	1.003	1.001	7.8
			300	1.030	1.009	1.023	1.002	1.001	7.8
	$\lambda_L=1,\lambda_X=1,$ $\nu_L=2,\nu_X=1$	0.416	100	1.037	0.976	1.101	2.116	1.038	32.7
			200	1.012	0.983	1.045	2.057	1.019	30.5
			300	1.005	0.987	1.020	2.036	1.018	27.1

		$c(\boldsymbol{\theta})$	n	$E(\hat{\alpha})$	$E(\hat{\lambda}_L)$	$E(\hat{\lambda}_X)$	$E(\hat{v}_L)$	$E(\hat{v}_X)$	AI
$\alpha=2$ $(\tau=0.5)$	$\lambda_L=2, \lambda_X=1,$ $v_L=1, v_X=1$	0.804	100	2.065	2.040	1.031	1.007	1.003	6.8
			200	2.037	2.017	1.009	1.003	1.004	6.1
			300	2.029	2.012	1.008	1.001	1.002	6.1
	$\lambda_L=1, \lambda_X=1,$ $v_L=1, v_X=1$	0.500	100	2.167	1.023	1.130	0.992	0.979	65.2
			200	2.086	1.014	1.052	0.993	0.990	8.7
			300	2.057	1.009	1.032	0.996	0.995	8.2
	$\lambda_L=1, \lambda_X=1,$ $v_L=2, v_X=1$	0.387	100	2.064	0.971	1.169	2.123	1.025	69.9
			200	2.020	0.986	1.050	2.044	1.018	54.2
			300	2.003	0.986	1.019	2.036	1.019	50.0
$\alpha=6$ $(\tau=0.75)$	$\lambda_L=2, \lambda_X=1,$ $v_L=1, v_X=1$	0.945	100	6.105	2.041	1.028	1.010	1.008	58.6
			200	6.073	2.014	1.020	1.003	1.004	41.2
			300	6.063	2.011	1.006	1.002	1.003	39.0
	$\lambda_L=1, \lambda_X=1,$ $v_L=1, v_X=1$	0.500	100	Un-convergence					
			200						
			300						
	$\lambda_L=1, \lambda_X=1,$ $v_L=2, v_X=1$	0.348	100	Un-convergence					
			200						
			300						

		$c(\boldsymbol{\theta})$	n	$E(\hat{\alpha})$	$E(\hat{\lambda}_L)$	$E(\hat{\lambda}_X)$	$E(\hat{\nu}_L)$	$E(\hat{\nu}_X)$	AI
$\alpha=2$ $(\tau=0.5)$	$\lambda_L=2, \lambda_X=1, \nu_L=1, \nu_X=1$	0.804	100	2.065	2.040	1.031	1.007	1.003	6.8
			200	2.037	2.017	1.009	1.003	1.004	6.1
			300	2.029	2.012	1.008	1.001	1.002	6.1
	$\lambda_L=1, \lambda_X=1, \nu_L=1, \nu_X=1$	0.500	100	2.167	1.023	1.130	0.992	0.979	65.2
			200	2.086	1.014	1.052	0.993	0.990	8.7
			300	2.057	1.009	1.032	0.996	0.995	8.2
	$\lambda_L=1, \lambda_X=1, \nu_L=2, \nu_X=1$	0.387	100	2.064	0.971	1.169	2.123	1.025	69.9
			200	2.020	0.986	1.050	2.044	1.018	54.2
			300	2.003	0.986	1.019	2.036	1.019	50.0
$\alpha=6$ $(\tau=0.75)$	$\lambda_L=2, \lambda_X=1, \nu_L=1, \nu_X=1$	0.945	100	6.105	2.041	1.028	1.010	1.008	58.6
			200	6.073	2.014	1.020	1.003	1.004	41.2
			300	6.063	2.011	1.006	1.002	1.003	39.0
	$\lambda_L=1, \lambda_X=1, \nu_L=1, \nu_X=1$	0.500	100			Un-convergence			
			200						
			300						
	$\lambda_L=1, \lambda_X=1, \nu_L=2, \nu_X=1$	0.348	100			Un-convergence			
			200						
			300						

Table 2(b) Simulation results under the Clayton copula with the Weibull margins based on 1000 repetitions. $\boldsymbol{\theta}=(\alpha,\lambda_L,\lambda_X,\nu_L,\nu_X)$.

		$c(\boldsymbol{\theta})$	n	$MSE(\hat{\alpha})$	$MSE(\hat{\lambda}_L)$	$MSE(\hat{\lambda}_X)$	$MSE(\hat{\nu}_L)$	$MSE(\hat{\nu}_X)$
$\alpha=0.5$ $(\tau=0.2)$	$\lambda_L=2,\lambda_X=1,$ $\nu_L=1,\nu_X=1$	0.700	100	0.0703	0.1456	0.0482	0.0096	0.0192
			200	0.0361	0.0655	0.0233	0.0042	0.0103
			300	0.0231	0.0438	0.0144	0.0027	0.0067
	$\lambda_L=1,\lambda_X=1,$ $\nu_L=1,\nu_X=1$	0.500	100	Un-convergence				
			200	0.0468	0.0356	0.0460	0.0048	0.0159
			300	0.0319	0.0227	0.0328	0.0034	0.0118
	$\lambda_L=1,\lambda_X=1,$ $\nu_L=2,\nu_X=1$	0.436	100	0.1502	0.0393	0.1964	0.0638	0.0544
			200	0.0858	0.0222	0.0830	0.0364	0.0296
			300	0.0598	0.0157	0.0555	0.0255	0.0191
$\alpha=1$ $(\tau=0.33)$	$\lambda_L=2,\lambda_X=1,$ $\nu_L=1,\nu_X=1$	0.736	100	0.1045	0.1111	0.0387	0.0078	0.0141
			200	0.0551	0.0494	0.0186	0.0035	0.0067
			300	0.0351	0.0319	0.0104	0.0022	0.0044
	$\lambda_L=1,\lambda_X=1,$ $\nu_L=1,\nu_X=1$	0.500	100	0.1469	0.0558	0.1186	0.0086	0.0259
			200	0.0688	0.0277	0.0505	0.0032	0.0138
			300	0.0423	0.0176	0.0338	0.0022	0.0096
	$\lambda_L=1,\lambda_X=1,$ $\nu_L=2,\nu_X=1$	0.416	100	0.4369	0.0522	0.3778	0.1298	0.0676
			200	0.1748	0.0237	0.1313	0.0575	0.0334
			300	0.1083	0.0143	0.0715	0.0317	0.0206

MSE decreasing when *n* increasing

		$c(\boldsymbol{\theta})$	n	$MSE(\hat{\alpha})$	$MSE(\hat{\lambda}_L)$	$MSE(\hat{\lambda}_X)$	$MSE(\hat{\nu}_L)$	$MSE(\hat{\nu}_X)$
$\alpha = 2$ ($\tau = 0.5$)	$\lambda_L = 2, \lambda_X = 1,$ $\nu_L = 1, \nu_X = 1$	0.804	100	0.1874	0.0796	0.0306	0.0064	0.0084
			200	0.0941	0.0360	0.0084	0.0029	0.0036
			300	0.0634	0.0253	0.0053	0.0020	0.0023
	$\lambda_L = 1, \lambda_X = 1,$ $\nu_L = 1, \nu_X = 1$	0.500	100	0.3592	0.0451	0.2617	0.0134	0.0244
			200	0.1281	0.0200	0.0631	0.0042	0.0113
			300	0.0752	0.0129	0.0319	0.0027	0.0072
	$\lambda_L = 1, \lambda_X = 1,$ $\nu_L = 2, \nu_X = 1$	0.387	100	1.0206	0.0597	0.6522	0.1845	0.0729
			200	0.3679	0.0216	0.1418	0.0546	0.0298
			300	0.2245	0.0140	0.0874	0.0363	0.0209
$\alpha = 6$ ($\tau = 0.75$)	$\lambda_L = 2, \lambda_X = 1,$ $\nu_L = 1, \nu_X = 1$	0.945	100	0.9622	0.0508	0.0846	0.0064	0.0067
			200	0.5022	0.0216	0.0694	0.0028	0.0039
			300	0.2841	0.0137	0.0033	0.0018	0.0018
	$\lambda_L = 1, \lambda_X = 1,$ $\nu_L = 1, \nu_X = 1$	0.500	100					
			200			Un-convergence		
			300					
	$\lambda_L = 1, \lambda_X = 1,$ $\nu_L = 2, \nu_X = 1$	0.348	100					
			200			Un-convergence		
			300					

MSE decreasing when *n* increasing

Coverage probability

- Formula of the standard error (SE) of is

$$SE(\hat{\theta}_j) = \sqrt{\{-\hat{H}^{-1}(\hat{\boldsymbol{\theta}})\}_{jj}} = \sqrt{\left[\left\{-\frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^{\mathrm{T}}}\ell_n(\hat{\boldsymbol{\theta}})\right\}^{-1}\right]_{jj}}$$

- The $(1-\beta)\%$ confidence interval (CI) for θ_j is

$$\hat{\theta}_j \pm Z_{\beta/2} \times SE(\hat{\theta}_j),$$

where Z_p is the p-th upper quantile for $N(0,1)$.

- $SD(\hat{\theta}_j) \equiv \sqrt{\frac{1}{R-1}\sum_{r=1}^{R}\{\hat{\theta}_{j(r)} - \bar{\hat{\theta}}_{j(\cdot)}\}^2}$

Table 3(a) Standard error estimates and coverage probabilities of the confidence intervals for α under the Clayton copula with Weibull margins based on 1000 repititions.

		$c(\boldsymbol{\theta})$	n	$SD(\hat{\alpha})$	$E\{SE(\hat{\alpha})\}$	Coverage probability $1-\beta=0.95$
$\alpha=0.5$ $(\tau=0.2)$	$\lambda_L=2, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.700	100	0.2646	0.2536	0.925
			200	0.1902	0.1833	0.946
			300	0.1521	0.1486	0.946
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.500	100	Un-convergence		
			200	0.2148	0.2061	0.936
			300	0.1777	0.1689	0.934
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=2, \nu_X=1$	0.436	100	0.3846	0.3573	0.938
			200	0.2921	0.2760	0.968
			300	0.2441	0.2273	0.923
$\alpha=1$ $(\tau=0.33)$	$\lambda_L=2, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.736	100	0.3206	0.3271	0.950
			200	0.2343	0.2283	0.947
			300	0.1870	0.1858	0.951
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.500	100	0.3750	0.3649	0.937
			200	0.2585	0.2552	0.940
			300	0.2037	0.2064	0.955
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=2, \nu_X=1$	0.416	100	0.6603	0.5268	0.949
			200	0.4181	0.3713	0.910
			300	0.3292	0.3015	0.930

		$c(\boldsymbol{\theta})$	n	$SD(\hat{\alpha})$	$E\{SE(\hat{\alpha})\}$	Coverage probability $1-\beta=0.95$
$\alpha=2$ ($\tau=0.5$)	$\lambda_L=2, \lambda_X=1, \nu_L=1, \nu_X=1$	0.804	100	0.4282	0.4479	0.954
			200	0.3047	0.3116	0.951
			300	0.2503	0.2538	0.949
	$\lambda_L=1, \lambda_X=1, \nu_L=1, \nu_X=1$	0.500	100	0.5758	0.5517	0.944
			200	0.3476	0.3514	0.946
			300	0.2684	0.2794	0.948
	$\lambda_L=1, \lambda_X=1, \nu_L=2, \nu_X=1$	0.387	100	1.0087	0.8939	0.921
			200	0.6065	0.5854	0.940
			300	0.4741	0.4564	0.934
$\alpha=6$ ($\tau=0.75$)	$\lambda_L=2, \lambda_X=1, \nu_L=1, \nu_X=1$	0.945	100	0.9758	0.9479	0.950
			200	0.7053	0.6671	0.945
			300	0.5295	0.5422	0.950
	$\lambda_L=1, \lambda_X=1, \nu_L=1, \nu_X=1$	0.500	100	Un-convergence		
			200			
			300			
	$\lambda_L=1, \lambda_X=1, \nu_L=2, \nu_X=1$	0.348	100	Un-convergence		
			200			
			300			

Table 3(b) Standard error estimates and coverage probabilities of the confidence intervals for λ_X under the Clayton copula with Weibull margins based on 1000 repitítions.

		$c(\boldsymbol{\theta})$	n	$SD(\hat{\lambda}_X)$	$E\{SE(\hat{\lambda}_X)\}$	Coverage probability $1-\beta=0.95$
$\alpha=0.5$ $(\tau=0.2)$	$\lambda_L=2, \lambda_X=1, \nu_L=1, \nu_X=1$	0.700	100	0.2175	0.1986	0.934
			200	0.1515	0.1423	0.943
			300	0.1191	0.1151	0.951
	$\lambda_L=1, \lambda_X=1, \nu_L=1, \nu_X=1$	0.500	100		Un-convergence	
			200	0.2141	0.1993	0.938
			300	0.1809	0.1639	0.940
	$\lambda_L=1, \lambda_X=1, \nu_L=2, \nu_X=1$	0.436	100	0.4415	0.3905	0.938
			200	0.2880	0.2750	0.938
			300	0.2355	0.2235	0.944
$\alpha=1$ $(\tau=0.33)$	$\lambda_L=2, \lambda_X=1, \nu_L=1, \nu_X=1$	0.736	100	0.1942	0.1821	0.943
			200	0.1353	0.1245	0.940
			300	0.1013	0.1001	0.942
	$\lambda_L=1, \lambda_X=1, \nu_L=1, \nu_X=1$	0.500	100	0.3412	0.2943	0.927
			200	0.2228	0.2074	0.934
			300	0.1826	0.1685	0.932
	$\lambda_L=1, \lambda_X=1, \nu_L=2, \nu_X=1$	0.416	100	0.6066	0.4712	0.927
			200	0.3597	0.3175	0.931
			300	0.2668	0.2498	0.943

		$c(\boldsymbol{\theta})$	n	$SD(\hat{\lambda}_X)$	$E\{SE(\hat{\lambda}_X)\}$	Coverage probability $1-\beta=0.95$
$\alpha=2$ $(\tau=0.5)$	$\lambda_L=2, \lambda_X=1, \nu_L=1, \nu_X=1$	0.804	100	0.1721	0.1366	0.946
			200	0.0912	0.0879	0.940
			300	0.0727	0.0706	0.945
	$\lambda_L=1, \lambda_X=1, \nu_L=1, \nu_X=1$	0.500	100	0.4950	0.3240	0.916
			200	0.2459	0.2032	0.923
			300	0.1757	0.1572	0.927
	$\lambda_L=1, \lambda_X=1, \nu_L=2, \nu_X=1$	0.387	100	0.7901	0.5642	0.921
			200	0.3734	0.3412	0.949
			300	0.2952	0.2623	0.959
$\alpha=6$ $(\tau=0.75)$	$\lambda_L=2, \lambda_X=1, \nu_L=1, \nu_X=1$	0.945	100	0.2897	0.1062	0.947
			200	0.2629	0.0758	0.961
			300	0.0576	0.0578	0.962
	$\lambda_L=1, \lambda_X=1, \nu_L=1, \nu_X=1$	0.500	100	Un-convergence		
			200			
			300			
	$\lambda_L=1, \lambda_X=1, \nu_L=2, \nu_X=1$	0.348	100	Un-convergence		
			200			
			300			

Table 3(c) Standard error estimates and coverage probabilities of the confidence intervals for ν_X under the Clayton copula with

Weibull margins based on 1000 repititions.

		$c(\boldsymbol{\theta})$	n	$SD(\hat{\nu}_X)$	$E\{SE(\hat{\nu}_X)\}$	Coverage probability $1-\beta=0.95$
$\alpha=0.5$ $(\tau=0.2)$	$\lambda_L=2, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.700	100	0.1384	0.1356	0.954
			200	0.1013	0.0986	0.956
			300	0.0818	0.0806	0.945
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.500	100	Un-convergence		
			200	0.1255	0.1214	0.945
			300	0.1079	0.1009	0.931
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=2, \nu_X=1$	0.436	100	0.2290	0.2240	0.934
			200	0.1697	0.1628	0.941
			300	0.1368	0.1337	0.943
$\alpha=1$ $(\tau=0.33)$	$\lambda_L=2, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.736	100	0.1187	0.1183	0.944
			200	0.0818	0.0833	0.955
			300	0.0664	0.0679	0.950
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.500	100	0.1607	0.1547	0.939
			200	0.1175	0.1137	0.937
			300	0.0978	0.0944	0.949
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=2, \nu_X=1$	0.416	100	0.2573	0.2378	0.916
			200	0.1814	0.1719	0.932
			300	0.1426	0.1409	0.946

		$c(\boldsymbol{\theta})$	n	$SD(\hat{\nu}_X)$	$E\{SE(\hat{\nu}_X)\}$	Coverage probability $1-\beta=0.95$
$\alpha=2$ ($\tau=0.5$)	$\lambda_L=2, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.804	100	0.0917	0.0893	0.945
			200	0.0597	0.0601	0.947
			300	0.0478	0.0483	0.951
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.500	100	0.1550	0.1394	0.913
			200	0.1060	0.0996	0.927
			300	0.0847	0.0809	0.944
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=2, \nu_X=1$	0.387	100	0.2689	0.2439	0.915
			200	0.1717	0.1744	0.934
			300	0.1433	0.1411	0.941
$\alpha=6$ ($\tau=0.75$)	$\lambda_L=2, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.945	100	0.0818	0.0732	0.943
			200	0.0621	0.0514	0.944
			300	0.0428	0.0418	0.941
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.500	100	Un-convergence		
			200			
			300			
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=2, \nu_X=1$	0.348	100	Un-convergence		
			200			
			300			

Table 3(d) Standard error estimates and coverage probabilities of the confidence intervals for $\mu_X = E(X)$ under the Clayton copula with Weibull margins based on 1000 repititions.

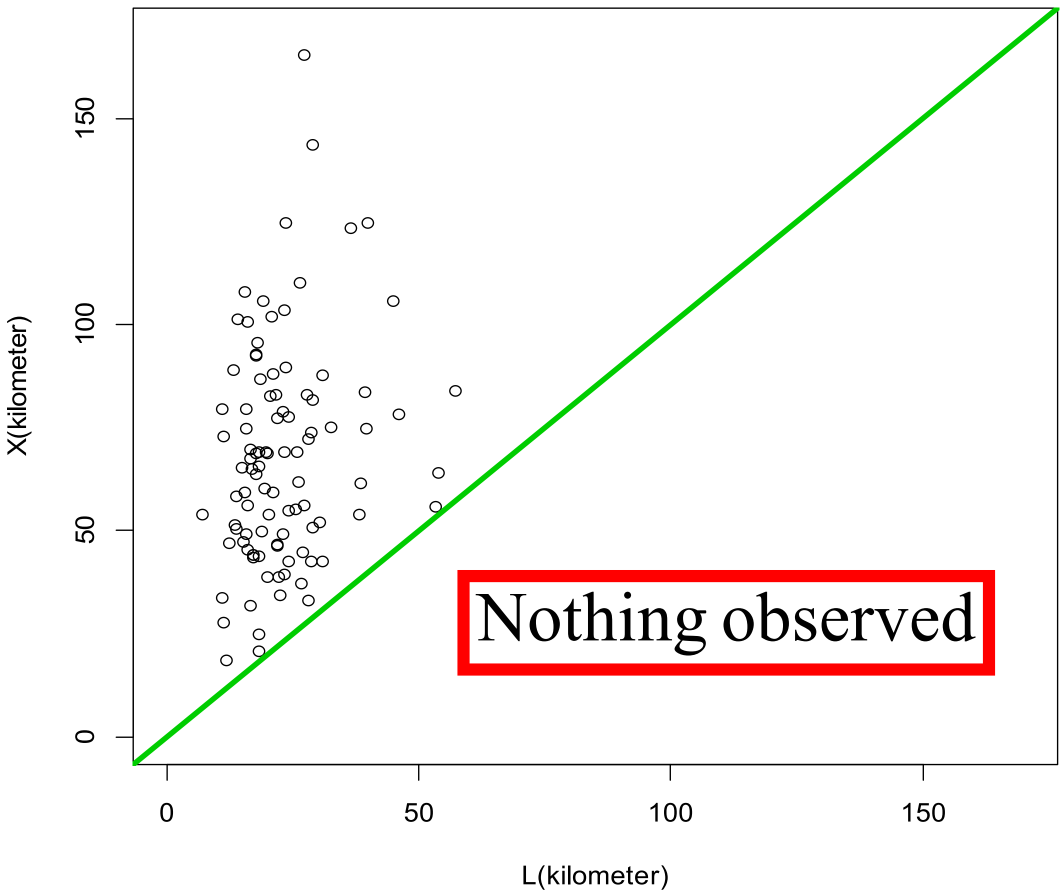
		$c(\boldsymbol{\theta})$	n	$SD(\hat{\mu}_X)$	$E\{SE(\hat{\mu}_X)\}$	Coverage probability $1-\beta=0.95$
$\alpha=0.5$ ($\tau=0.2$)	$\lambda_L=2, \lambda_X=1, \nu_L=1, \nu_X=1$	0.700	100	0.1526	0.1453	0.937
			200	0.1102	0.1055	0.938
			300	0.0878	0.0861	0.940
	$\lambda_L=1, \lambda_X=1, \nu_L=1, \nu_X=1$	0.500	100		Un-convergence	
			200	0.1548	0.1468	0.941
				0.1331	0.1218	0.931
	$\lambda_L=1, \lambda_X=1, \nu_L=2, \nu_X=1$	0.436	100	0.2744	0.2637	0.945
			200	0.2017	0.1968	0.942
			300	0.1681	0.1629	0.950
$\alpha=1$ ($\tau=0.33$)	$\lambda_L=2, \lambda_X=1, \nu_L=1, \nu_X=1$	0.736	100	0.1379	0.1369	0.942
			200	0.1013	0.0961	0.940
			300	0.0781	0.0782	0.950
	$\lambda_L=1, \lambda_X=1, \nu_L=1, \nu_X=1$	0.500	100	0.2167	0.2026	0.929
			200	0.1571	0.1506	0.936
			300	0.1314	0.1246	0.944
	$\lambda_L=1, \lambda_X=1, \nu_L=2, \nu_X=1$	0.416	100	0.3166	0.2874	0.928
			200	0.2308	0.2172	0.934
			300	0.1845	0.1791	0.943

		$c(\boldsymbol{\theta})$	n	$SD(\hat{\mu}_X)$	$E\{SE(\hat{\mu}_X)\}$	Coverage probability $1-\beta=0.95$
$\alpha=2$ ($\tau=0.5$)	$\lambda_L=2, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.804	100	0.1171	0.1116	0.947
			200	0.0772	0.0764	0.948
			300	0.0621	0.0618	0.960
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.500	100	0.2318	0.1984	0.916
			200	0.1569	0.1446	0.928
			300	0.1228	0.1175	0.941
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=2, \nu_X=1$	0.387	100	0.3439	0.3076	0.919
			200	0.2323	0.2301	0.952
			300	0.1899	0.1874	0.958
$\alpha=6$ ($\tau=0.75$)	$\lambda_L=2, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.945	100	0.1050	0.0921	0.935
			200	0.0788	0.0654	0.939
			300	0.0537	0.0536	0.951
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=1, \nu_X=1$	0.500	100	Un-convergence		
			200			
			300			
	$\lambda_L=1, \lambda_X=1,$ $\nu_L=2, \nu_X=1$	0.348	100	Un-convergence		
			200			
			300			

Data analysis

- The lifetimes of brake pads of automobiles (Kalbfleisch and Lawless 1992).
  - X: The number of kilometers of the car driven at failure (Our interest)
  - L : The number of kilometers of the car driven at the study start
  - sample size : 98

Data analysis



Data analysis

- Model:
  - M_1: The Clayton copula with exponential margins
  - M_2: The Clayton copula with Weibull margins

- Model selection:
  - Akaike information criterion (AIC) (Akaike, 1973)
  - Bayesian information criterion (BIC) (Schwarz, 1978)

Data analysis

- Akaike information criterion (AIC):

$$AIC = -2\log L + 2k$$

- Bayesian information criterion (BIC):

$$BIC = -2\log L + k\log n$$

- k : The number of unknown parameters
- n : The sample size

Data analysis

Table 4 The MLE for parameters, maximized value of the log-likelihood function, AIC and BIC.

Model	$\hat{E}(X)$	$\hat{\alpha}$	$\hat{\lambda}_L$	$\hat{\lambda}_X$	$\hat{\nu}_L$	$\hat{\nu}_X$	$\log L$	AIC	BIC
M_1	48.31	0.924	0.0364	0.0207	1 (fixed)	1 (fixed)	-874.31	1754.61	1762.36
	(5.49)	(0.663)	(0.0081)	(0.0024)	-	-	-	-	-
M_2	64.82	0.332	2.89×10^{-4}	3.09×10^{-5}	2.493	2.419	-806.62	1623.24	1636.17
	(3.25)	(0.358)	(1.86×10^{-4})	(3.15×10^{-5})	(0.183)	(0.222)	-	-	-

M_1: The Clayton copula with exponential margins.

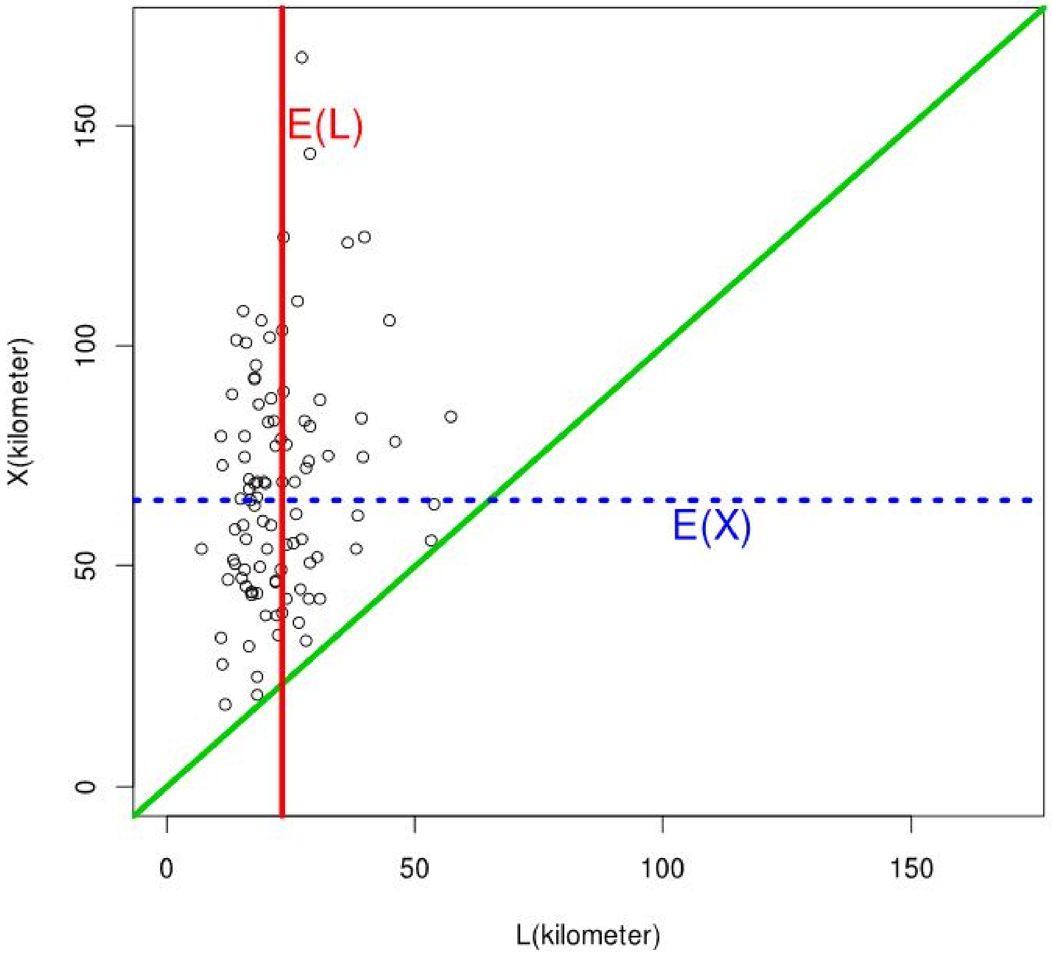
M_2: The Clayton copula with Weibull margins.

$(\cdot)$: The standard error (SE) of each parameters.

Smaller

- M_2 is more suitable than M_1.

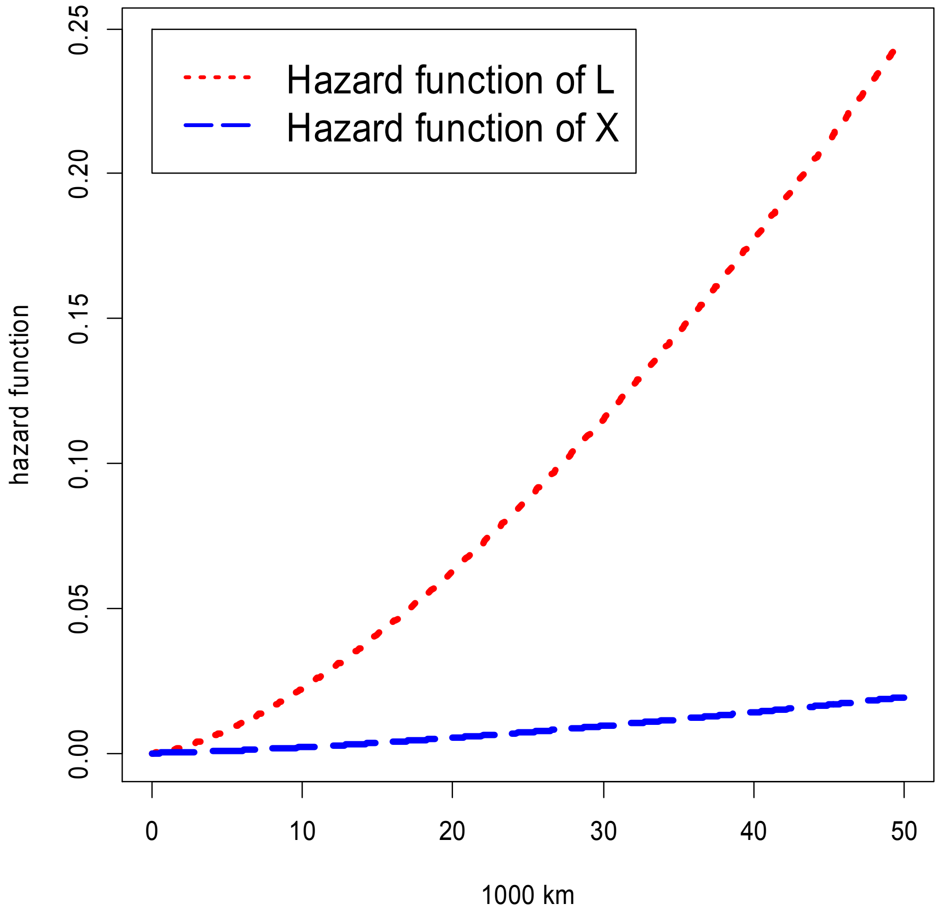
Data analysis

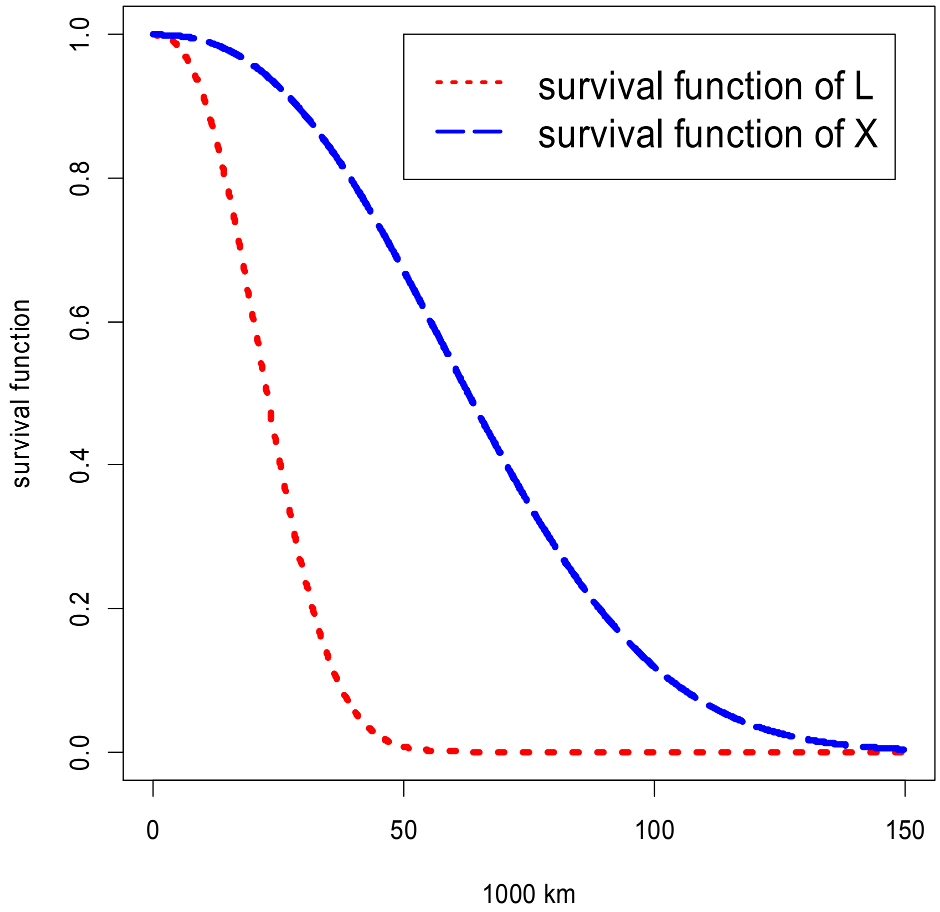


$$\hat{E}(X) = \Gamma(1 + 1/\hat{v}_X) / \hat{\lambda}_X^{\ 1/\hat{v}_X} = 64.82$$

$$\hat{E}(L) = \Gamma(1 + 1/\hat{v}_L) / \hat{\lambda}_L^{\ 1/\hat{v}_L} = 23.33$$

Hazard functions
Survival functions





Data analysis

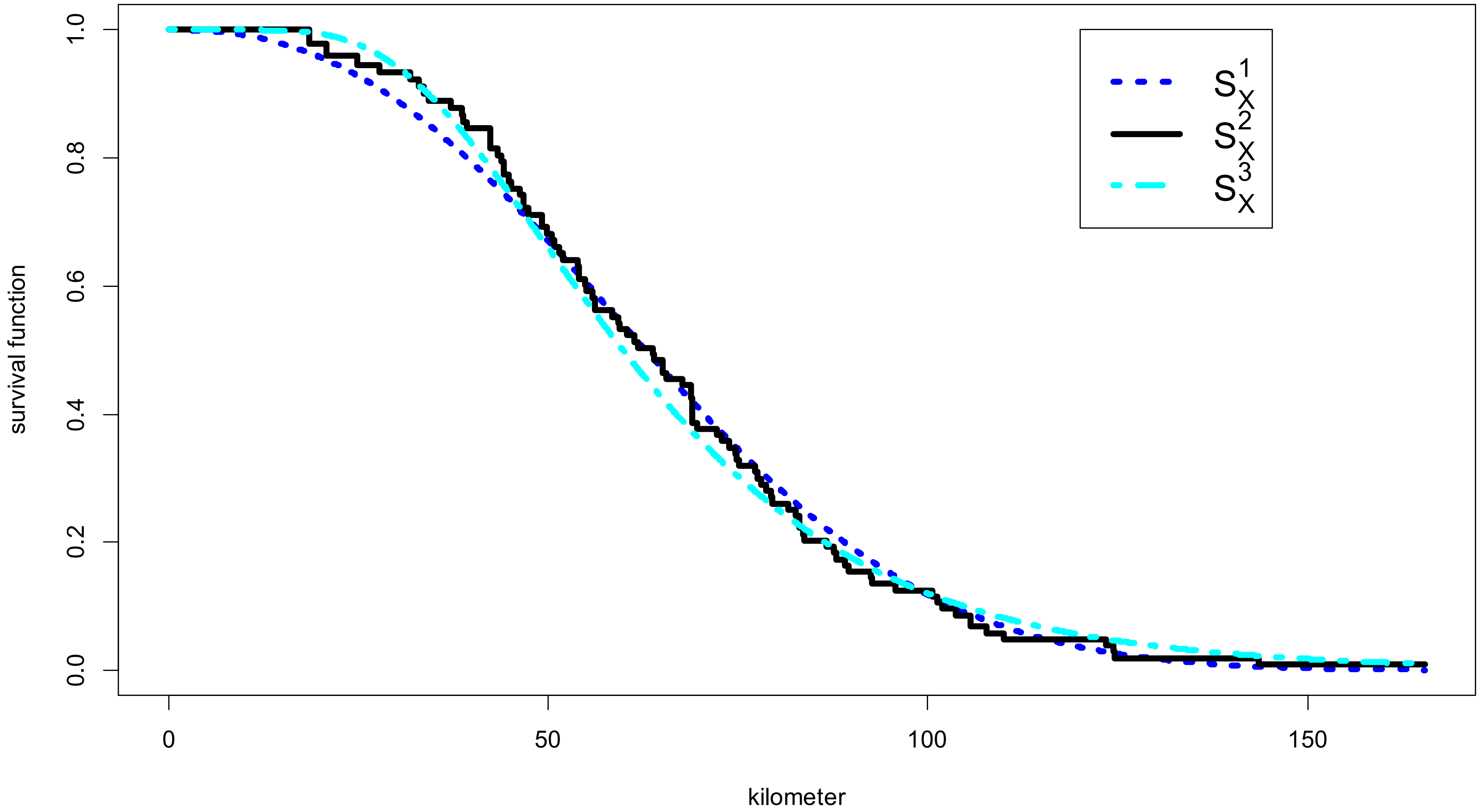
- $S_X^1(x) = \exp(-\hat{\lambda}_X x^{\hat{\nu}_X})$ (our method)
- $S_X^2(x)$ compute by using R-function "EMURA.Clayton"

  (Emura and Murotani 2015)

  ("depend.truncation" package)
- $S_X^3(x) = 1 - \Phi[\{\log(x) - \hat{\mu}\}/\hat{\sigma}]$, where $\Phi(\cdot)$ is distribution of $N(0,1)$, $\hat{\mu} = 4.092$ and $\hat{\sigma} = 0.4368$

  (Kalbfleisch and Lawless 1992)

Data analysis



Summary

- New results for the formulas of the inclusion probability $c(\boldsymbol{\theta})$ is obtained under the copula model.
- We propose the randomized-Newton-Raphson algorithm for maximizing the log-likelihood.
- In real data, we found that there is weak positive dependence between two variables. This implies that the results of Kalbfleisch and Lawless (1992) that assumed independence are questionable.

Future work

- Extension of the Weibull models to generalized gamma models which are common in industrial applications. (Fan and Yu 2013)

- Extension of our models under the presence of regressors.(Chen 2010; Emura and Chen 2014)

Reference

- Akaike H (1973) Information theory and an extension of the maximum likelihood principle, Petrov BN and Csaki F, *Proc. 2nd International Symposium on Information Theory*, Akademiai Kiado, Budapest, pp.267-281.
- Chen YH (2010) Semiparametric marginal regression analysis for dependent competing risks under an assumed copula. *Journal of the Royal Statistical Society,* Ser. B 72: 235-51.
- Emura T, Wang W (2010) Testing quasi-independence for truncation data. *Journal of Multivariate Analysis* 101: 223-239.
- Emura T, Konno Y (2012) Multivariate normal distribution approaches for dependently truncated data. *Statistical Papers* 53:133-149.
- Emura T, Chen YH (2014) Gene selection for survival data under dependent censoring: a copula-based approach. *Statistical Methods in Medical Research* DOI: 10.1177/0962280214533378.
- Emura T, Murotani K (2015) An algorithm for estimating survival under a copula-based dependent truncation model. *TEST* DOI: 10.1007/s11749-015-0432-8.
- Fan TH, Yu CH (2013) Statistical inference on contant stress accelerated life tests under generalized gamma lifetime distributions. *Quality and Reliability Engineering International* 29: 631-638.
- Hu YH, Emura T (2015) Maximum likelihood estimation for a special exponential family under random double-truncation. *Computational Statistics* DOI: 10.1007/s00180-015-0564-z.
- Kalbfleisch JD, Lawless JF (1992) Some useful statistical methods for truncated data. *Journal of quality technology* 24: 145-152.
- Lakhal-Chaieb L, Rivest LP, Abdous B (2006) Estimating survival under a dependent truncation. *Biometrika* 93: 665–669.
- Lynden-Bell D (1971) A method of allowing for known observational selection in small samples applied to 3RC quasars. *Mon Not R Astron Soc* 155: 95-118.

Thank you for your listening