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Nonparametric Procedures for Comparing Umbrella Pattern Treatment Effects with a Control in a One-Way Layout

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SUMMARY

In this paper we are concerned with comparing umbrella pattern treatment effects with a control in a one-way layout. The problem of testing whether there is at least one treatment that is better than the control is considered. Distribution-free tests are proposed for both cases where the peak of the umbrella is known or unknown. Approximate small-sample critical values are presented and the results of a Monte Carlo power study are discussed.

1. Introduction

Nonparametric procedures for comparing several treatments with a control in a one-way layout have been extensively studied. For example, Steel (1959) proposed a multiple-comparison rank sum test based on pairwise rankings for comparing several treatments with a control, whereas Dunn (1964) suggested a treatment versus control rank sum test based on joint rankings. In their procedures, however, they did not use any prior information about the pattern of treatment effects. Shirley (1977) considered a nonparametric version of Williams' (1971, 1972) test for comparing increasing doses of a substance with a control. Her procedure employs the prior information that if there were a response to the substance the treatment effects would be monotonically ordered. Moreover, Shirley's test can be used to determine the lowest dose level at which there is evidence of a difference from the control.

In a drug study, for instance, increasing dosage levels may be compared with a zero-dose control. Suppose the investigator believes that if the treatment effects are not identical to the control, then, in general, the higher the dose of the drug applied, the better (say, higher) will be the resulting treatment effect. However, it is also known that the subject may actually succumb to toxic effects at high doses, thereby decreasing the treatment effects. In this case, an ordering in the treatment effects that is monotonically increasing up to a point, followed by a monotonic decrease is anticipated. Since this corresponds to an up–down ordering of the treatment effects, they are said to follow an umbrella pattern [see, for example, Mack and Wolfe (1981)]. The point that separates the treatment effects into the two different ordering groups is called the peak of the umbrella. To compare several treatments with a control in such a setting, test procedures utilizing this information about an umbrella pattern alternative would be preferred.

In this paper we are concerned with the problem of testing whether there is at least one treatment that is better than the control when the prior information about the umbrella pattern treatment effects is available. Suppose that $X_{i1}, \ldots, X_{in_i}$ $(i = 0, 1, \ldots, k)$ are $k + 1$ independent random samples from populations with continuous distribution functions $F_i(x) = F(x - \theta_i)$ $(i = 0, 1, \ldots, k)$, respectively. The zero population $(i = 0)$ is the control and the other k populations are treatments. Specifically, we consider testing the null hypothesis H_0: $[\theta_0 = \theta_1 = \cdots = \theta_k]$ against the alternative hypothesis H_A: $[\theta_i > \theta_0$ for at least one $i]$. In addition, we assume that, under H_A, $\theta_1 \leqslant \cdots \leqslant \theta_p \geqslant \cdots \geqslant \theta_k$, for some p.

Key words: Monte Carlo power study; Nonparametric treatments versus control procedure; One-way layout; Ordered treatment effects; Umbrella pattern treatment effects; Williams' test.

The problem of testing whether treatment effects follow an umbrella pattern has been considered by Mack and Wolfe (1981), Simpson and Margolin (1986), and Chen and Wolfe (1990), among others. Note that tests for umbrella alternatives are also applicable to the problem considered in this paper since the alternative H_A of interest here can be regarded as a special case of umbrella alternatives. In comparing several treatments with a control, however, we are usually more interested in multiple comparison procedures that can be used to decide which treatments (if any) are better than the control.

In Section 2 we propose a distribution-free test for comparing umbrella pattern treatment effects with a control when the peak of the umbrella is known a priori. The estimation of the lowest dose that is more effective than the control and the highest dose that is still better than the control is also discussed. In Section 3 we discuss two distribution-free tests for the unknown peak setting based on the different methods for estimating the umbrella peak suggested by Mack and Wolfe (1981) and Simpson and Margolin (1986), respectively. In Section 4 we present approximate small-sample critical values for these test procedures. In Section 5 an illustrative example involving Ames Salmonella/microsome test data is provided. Section 6 presents the results of a Monte Carlo simulation investigation of the relative powers of several competing tests for a variety of umbrella pattern treatment effects configurations.

2. Case of Known Umbrella Peak

Let R_{ij} be the rank of X_{ij} among the $N = \sum_{i=1}^{k} n_i$ observations and let $\bar{R}_i = \sum_{j=1}^{n_i} R_{ij}/n_i$ be the average rank of the ith sample, $i = 0, 1, \ldots, k$. Suppose that, under H_A, the peak of the umbrella is known to be at group p $(1 \leq p \leq k)$. Furthermore, assume that $n_0 = c$ and $n_1 = \cdots = n_k = n$. Let $\hat{R}_1 \leq \cdots \leq \hat{R}_p \geq \cdots \geq \hat{R}_k$ be the isotonic regression of $\bar{R}_1, \ldots, \bar{R}_k$ under the restriction $\theta_1 \leq \cdots \leq \theta_p \geq \cdots \geq \theta_k$. [For a discussion of the algorithm for obtaining $\hat{R}_1, \ldots, \hat{R}_k$, see Chen and Wolfe (1990).] Since, under H_A, $\theta_p > \theta_0$ is equivalent to $\theta_i > \theta_0$ for some i, we propose to reject H_0 for large values of

$$T_p = (\hat{R}_p - \bar{R}_0)[\{N(N+1)/12\}(1/n + 1/c)]^{-1/2}. \tag{1}$$

From the derivation of $\hat{R}_p$, we note that

$$\hat{R}_p = \max_{1 \leq u \leq p \leq v \leq k} \sum_{i=u}^{v} \bar{R}_i/(v - u + 1). \tag{2}$$

Therefore, the statistic T_p becomes

$$T_p = \max_{1 \leq u \leq p \leq v \leq k} \left\{ \sum_{i=u}^{v} (\bar{R}_i - \bar{R}_0)/(v - u + 1) \right\} [\{N(N+1)/12\}(1/n + 1/c)]^{-1/2}. \tag{3}$$

In particular, the test based on T_k is Shirley's (1977) test for comparing ordered treatment effects with a control. Moreover, suppose that $N \to \infty$ in such a way that $n/(n + c) \to \rho$, with $0 < \rho < 1$. From the results of Miller (1966), we know that, under H_0, the statistic T_p converges in distribution to the statistic

$$Y_p = \max_{1 \leq u \leq p \leq v \leq k} \sum_{i=u}^{v} W_i/(v - u + 1) \tag{4}$$

as $N \to \infty$, where the random vector $(W_1, \ldots, W_k)$ has a multivariate normal distribution with $E(W_i) = 0$, $\text{var}(W_i) = 1$, and $\text{cov}(W_i, W_j) = \rho$, $i \neq j = 1, \ldots, k$.

If the test based on T_p rejects H_0, one would wish to determine which dosage levels are more effective than the control. To answer this question, let $t_p(\alpha; n, c, k)$ be the value such that

$$\Pr\{T_p \geq t_p(\alpha; n, c, k) \mid H_0\} = \alpha.$$

We then decide that $\theta_i > \theta_0$ for $u \leq i \leq v$, where $1 \leq u \leq p \leq v \leq k$, if

$$\hat{R}_u - \bar{R}_0 \geq t_p(\alpha; n, c, k)[\{N(N+1)/12\}(1/n + 1/c)]^{1/2}$$

and

$$\hat{R}_v - \bar{R}_0 \geq t_p(\alpha; n, c, k)[\{N(N+1)/12\}(1/n + 1/c)]^{1/2}. \tag{5}$$

It is obvious that the Type I error rate for this procedure is controlled since

$$\begin{aligned} \alpha &= \Pr\{\hat{R}_p - \bar{R}_0 \geq t_p(\alpha; n, c, k)[\{N(N+1)/12\}(1/n + 1/c)]^{1/2} \mid H_0\} \\ &\geq \Pr\{\hat{R}_i - \bar{R}_0 \geq t_p(\alpha; n, c, k)[\{N(N+1)/12\}(1/n + 1/c)]^{1/2}, \; i = 1, \ldots, k \mid H_0\}. \end{aligned}$$

Note that when ties occur in the rankings, a modification of the test based on T_p is obtained by replacing $N(N+1)/12$ with $N(N+1)/12 - \sum_{g\in G}(t_g^3 - t_g)/(12(N-1))$, where G is the set of groups of ties and t_g is the number of observations tied in the gth group.

3. Case of Unknown Umbrella Peak

If, under H_A, one expects that the peak group of the umbrella is relatively close to the kth group, then the method suggested by Simpson and Margolin (1986) can be utilized to estimate the unknown peak. Let U_{ij} be the usual Mann–Whitney statistic corresponding to the number of observations in sample j that exceed observations in sample i and let $Q_j = \sum_{i=1}^{j-1} U_{ij}$, $j = 2, \ldots, k$. Set $\hat{p}_s = \max_{2\leq j\leq k}\{j\colon Q_j \geq (j-1)n^2/2\}$. We then reject H_0 for large values of

$$T_{\hat{p}_s} = \max_{1\leq u\leq \hat{p}_s\leq v\leq k}\left\{\sum_{i=u}^{v}(\bar{R}_i - \bar{R}_0)/(v-u+1)\right\}[\{N(N+1)/12\}(1/n+1/c)]^{-1/2}. \tag{6}$$

For the more general setting in which no information concerning the location of the peak group is available, let $Z_j = \sum_{i=1}^{j-1} U_{ij} + \sum_{i=j+1}^{k} U_{ij}$, $j = 1, \ldots, k$. Following the suggestions of Mack and Wolfe (1981), we first choose the group $\hat{p}_m$ such that $Z_{\hat{p}_m} = \max\{Z_j, j = 1, \ldots, k\}$. The null hypothesis H_0 is then rejected for large values of

$$T_{\hat{p}_m} = \max_{1\leq u\leq \hat{p}_m\leq v\leq k}\left\{\sum_{i=u}^{v}(\bar{R}_i - \bar{R}_0)/(v-u+1)\right\}[\{N(N+1)/12\}(1/n+1/c)]^{-1/2}. \tag{7}$$

It is noted in Mack and Wolfe (1981) that there is a positive probability to observe two or more groups tied for the largest Z_j sample values. In this situation, let χ be the set of groups tied for the maximum Z_j. We then take the value of $T_{\hat{p}_m}$ to be the average of the T_j's for those j in the set χ.

Let $t_{\hat{p}_s}(\alpha; n, c, k)$ and $t_{\hat{p}_m}(\alpha; n, c, k)$ be the upper αth percentiles of the null distributions of $T_{\hat{p}_s}$ and $T_{\hat{p}_m}$, respectively. If the test based on $T_{\hat{p}_s}$ (or $T_{\hat{p}_m}$) rejects H_0, a level $(1-\alpha)$ multiple comparison procedure similar to that described in Section 2, but employing the critical value $t_{\hat{p}_s}(\alpha; n, c, k)$ [or $t_{\hat{p}_m}(\alpha; n, c, k)$], can be used to determine which treatments are significantly better than the control. Moreover, when ties occur in the rankings, the procedures based on $T_{\hat{p}_s}$ and $T_{\hat{p}_m}$ can be modified by applying the corrections for ties in the rankings mentioned in Section 2.

4. Small-Sample Null Distribution of T_p and $T_{\hat{p}_s}$

In general, the null distribution of a nonparametric test statistic can be computed by evaluating the statistic for every possible arrangement of the appropriate ranks. However, the required number of arrangements becomes prohibitively large very rapidly as each of the sample sizes $n_0, n_1, \ldots, n_k$ gets large. In order to obtain approximate critical values for the tests based on T_p and $T_{\hat{p}_s}$, we simulated the null distributions for number of treatments $k = 3, 4, 5$ and for equal sample sizes $n_0 = n_1 = \cdots = n_k = n = 3(1)10$.

Each of these simulated distributions was based on 10,000 replications. Therefore, we are guaranteed a standard error no greater than .003 for estimating tail probabilities at least as small as .10. (In fact, the standard error is even less for smaller tail probabilities such as .05 or .01.) The necessary uniformly distributed random numbers in (0, 1] were generated by the International Mathematical and Statistical Libraries (IMSL) routine RNUN. Note that the statistics T_i and T_{k-i+1} $(i = 1, \ldots, k)$ have the same distribution under the null hypothesis H_0. Therefore, we simply simulated the null distributions of T_p for $p = [k/2] + 1, \ldots, k$, where $[k/2]$ is the greatest integer less than or equal to $k/2$. Approximate critical values for the null distributions for T_p and $T_{\hat{p}_s}$ corresponding to levels closest to .01, .05, and .10 are presented in Tables 1, 2, 3, and 4.

5. An Example

In in vitro mutagenicity assays, experimental organisms may succumb to toxic effects at high doses of the test agent, thereby reducing the number of organisms at risk of mutation and causing a downturn in the dose–response curve (Margolin, Kaplan, and Zeiger, 1981). The data in Table 5 are the numbers of visible revertant colonies observed on plates containing Salmonella bacteria of strain TA98 and exposed to various doses of Acid Red 114. [These data correspond to the third replication of the Ames test results as given in Simpson and Margolin (1986).]

For testing whether there is at least one treatment that is better than the zero-dose control, $T_{\hat{p}_s}$ is applied to the data of Table 5. First we find the estimated peak group to be $\hat{p}_s = 3$ (1,000 μg/ml). Second, we compute the average ranks, obtaining $\bar{R}_0 = 5.8$, $\bar{R}_1 = 8$, $\bar{R}_2 = 13.7$, $\bar{R}_3 = 16.8$, $\bar{R}_4 = 10$, and $\bar{R}_5 = 2.7$. Note that for these data the correction for ties is

$$\{4(2^3 - 2)\}/\{(12)(17)\} = .1176.$$

Table 1
Approximate critical values for T_p when $n_0 = n_1 = \cdots = n_3 = n$

		p = Peak of the umbrella	
n	Level	2	3
3	.01	2.265	2.265
	.05	1.812	1.755
	.10	1.472	1.472
4	.01	2.339	2.302
	.05	1.782	1.745
	.10	1.485	1.448
5	.01	2.316	2.352
	.05	1.791	1.764
	.10	1.497	1.461
6	.01	2.368	2.266
	.05	1.756	1.735
	.10	1.449	1.409
7	.01	2.388	2.339
	.05	1.755	1.755
	.10	1.451	1.408
8	.01	2.345	2.372
	.05	1.741	1.746
	.10	1.452	1.426
9	.01	2.327	2.312
	.05	1.756	1.734
	.10	1.443	1.417
10	.01	2.362	2.334
	.05	1.760	1.741
	.10	1.444	1.425

Table 2
Approximate critical values for T_p when $n_0 = n_1 = \cdots = n_4 = n$

		p = Peak of the umbrella	
n	Level	3	4
3	.01	2.282	2.237
	.05	1.765	1.765
	.10	1.483	1.461
4	.01	2.331	2.331
	.05	1.813	1.793
	.10	1.494	1.454
5	.01	2.342	2.299
	.05	1.826	1.772
	.10	1.504	1.439
6	.01	2.361	2.328
	.05	1.749	1.749
	.10	1.459	1.421
7	.01	2.374	2.347
	.05	1.787	1.761
	.10	1.469	1.422
8	.01	2.374	2.324
	.05	1.775	1.738
	.10	1.461	1.433
9	.01	2.333	2.321
	.05	1.783	1.750
	.10	1.463	1.436
10	.01	2.408	2.355
	.05	1.818	1.769
	.10	1.473	1.452

Table 3
Approximate critical values for T_p when $n_0 = n_1 = \cdots = n_5 = n$

		p = Peak of the umbrella		
n	Level	3	4	5
3	.01	2.294	2.294	2.256
	.05	1.810	1.835	1.784
	.10	1.504	1.504	1.472
4	.01	2.325	2.313	2.283
	.05	1.783	1.800	1.763
	.10	1.517	1.475	1.463
5	.01	2.275	2.335	2.317
	.05	1.760	1.769	1.733
	.10	1.473	1.473	1.437
6	.01	2.366	2.329	2.329
	.05	1.799	1.781	1.713
	.10	1.489	1.480	1.416
7	.01	2.364	2.342	2.314
	.05	1.772	1.786	1.748
	.10	1.489	1.487	1.422
8	.01	2.357	2.366	2.321
	.05	1.774	1.772	1.732
	.10	1.476	1.464	1.414
9	.01	2.367	2.357	2.317
	.05	1.790	1.803	1.735
	.10	1.501	1.498	1.427
10	.01	2.411	2.394	2.343
	.05	1.827	1.812	1.752
	.10	1.494	1.495	1.444

Table 4
Approximate critical values for $T_{\hat{p}_s}$ when $n_0 = n_1 = \cdots = n_k = n$

		k = No. of treatments		
n	Level	3	4	5
3	.01	2.378	2.419	2.524
	.05	1.982	2.008	2.027
	.10	1.642	1.735	1.759
4	.01	2.488	2.540	2.650
	.05	1.968	2.017	2.025
	.10	1.671	1.713	1.733
5	.01	2.486	2.578	2.550
	.05	1.951	2.019	2.030
	.10	1.630	1.719	1.724
6	.01	2.490	2.591	2.576
	.05	1.898	2.000	2.041
	.10	1.592	1.689	1.726
7	.01	2.567	2.543	2.582
	.05	1.933	2.021	2.037
	.10	1.625	1.708	1.728
8	.01	2.492	2.588	2.607
	.05	1.932	2.010	2.027
	.10	1.612	1.700	1.714
9	.01	2.461	2.566	2.592
	.05	1.913	2.010	2.060
	.10	1.588	1.705	1.745
10	.01	2.474	2.592	2.599
	.05	1.922	2.063	2.074
	.10	1.616	1.718	1.754

Table 5
Revertant colonies for Acid Red 114, TA98, *hamster liver activation*

Dose (μg/ml)					
0	100	333	1,000	3,333	10,000
23	27	28	41	28	16
22	23	37	37	21	19
14	21	35	43	30	13

Therefore, we obtain

$$T_{\hat{p}_s} = (11)[\{(18)(19)/12 - .1176\}(2/3)]^{-1/2} = 11/4.3499 = 2.529.$$

With $k = 5$ and $n_0 = n_1 = \cdots = n_5 = 3$, we find from Table 4 that the approximate 1% and 10% significance critical values for $T_{\hat{p}_s}$ are 2.524 and 1.759, respectively. Thus there is a significant treatment effect at the high dose levels. Furthermore, since

$$\hat{R}_{\hat{p}_s-1} - \bar{R}_0 = 7.9 > (1.759)(4.3499) = 7.65,$$

$$\hat{R}_{\hat{p}_s-2} - \bar{R}_0 = 2.2 < 7.65,$$

and

$$\hat{R}_{\hat{p}_s+1} - \bar{R}_0 = 4.2 < 7.65,$$

we conclude, at the 10% significance level, that the dosages between 333 μg/ml and 1,000 μg/ml, inclusive, are more effective than the zero-dose control.

6. Monte Carlo Power Study

We conducted a Monte Carlo study to examine the relative powers of eight competing distribution-free test procedures based on joint rankings for comparing general umbrella pattern treatment effects with a control, namely, Dunn's (1964) test, D, for comparing general treatment effects with a control; Shirley's (1977) test, S, for comparing ordered treatment effects with a control; the Mack–Wolfe (1981) tests, A_p and $A_{\hat{p}_m}$, for umbrella alternatives with known and unknown peak, respectively; the Simpson–Margolin (1986) test, $S_M(\frac{1}{2})$, for umbrella alternatives when the umbrella peak is expected to be relatively close to the kth group; and the tests based on T_p, $T_{\hat{p}_s}$, and $T_{\hat{p}_m}$, respectively. The study was performed for $k = 3$, 4, and 5 populations, with $n_0 = n_1 = \cdots = n_k = 5$ observations per sample in each case, and for a variety of different umbrella pattern treatment effects.

For each of these settings, appropriate normal and exponential deviates were derived by the IMSL routines RNNOR and RNEXP, respectively. In each case, we used 10,000 replications in obtaining the various power estimates. Exact critical values were used, when available, in the sample rejection counts; otherwise, simulated critical values were used. To make the power comparisons meaningful, we employed randomization to achieve the nominal levels of $\alpha = .05$ or .10. The simulated power estimates for the eight tests considered in the study are presented in Tables 6, 7, and 8. The designated alternative configurations correspond to values of $\theta_1 - \theta_0, \ldots, \theta_k - \theta_0$.

We observe from the simulation results that Shirley's test has excellent power when the treatment effects have a monotonic ordering. Likewise, the test based on T_p provides excellent power against umbrella pattern treatment effects when the peak is correctly chosen. This is not surprising since both tests are designed to detect for their respective special classes of alternatives. From Tables 6, 7, and 8, however, we also see that the power of Shirley's test drops sharply when there is a downturn in the umbrella. Similarly, we would expect the power of the test based on T_p to decline when the peak is incorrectly selected.

In general, the statistic $T_{\hat{p}_s}$ provides a better test than does either $T_{\hat{p}_m}$ or D for the unknown peak setting when the peak group is relatively close to the kth population. When, however, the location of the peak group is relatively far from the kth population, the test based on $T_{\hat{p}_s}$ performs poorly. In these cases, the tests based on $T_{\hat{p}_m}$ and D, respectively, are both superior to the one based on $T_{\hat{p}_s}$.

Note that the power performance of the test based on $T_{\hat{p}_m}$ is similar to that of the test based on D for comparing general umbrella pattern treatment effects with a control. This is not a surprise since, in the case of $n_1 = \cdots = n_k = n$, the choice of $\hat{p}_m$ is in fact to select $\hat{p}_m$ such that $\bar{R}_{\hat{p}_m} = \max\{\bar{R}_j, j = 1, \ldots, k\}$. According to the algorithm for deriving the isotonic regression of $\bar{R}_1, \ldots, \bar{R}_k$ under an umbrella pattern restriction, we obtain $\hat{R}_{\hat{p}_m} = \bar{R}_{\hat{p}_m}$. Therefore, the test based on $T_{\hat{p}_m}$ is actually equivalent to Dunn's (1964) test for testing H_0 against H_A. (The slight differences in the estimated

Table 6
Monte Carlo power estimates for $k = 3$ and $n_0 = n_1 = n_2 = n_3 = 5$

$\theta_1 - \theta_0$	$\theta_2 - \theta_0$	$\theta_3 - \theta_0$	α	T_p	S	$T_{\hat{p}_s}$	$T_{\hat{p}_m}$	D	A_p	$S_M(\frac{1}{2})$	$A_{\hat{p}_m}$
					a. Normal						
0	0	1	.05	.366	.366	.291	.239	.242	.382	.279	.168
			.10	.504	.504	.431	.383	.389	.534	.428	.288
0	.5	1	.05	.397	.397	.326	.282	.287	.507	.396	.254
			.10	.540	.540	.476	.434	.430	.648	.547	.391
.5	.5	1	.05	.414	.414	.352	.329	.332	.416	.322	.199
			.10	.557	.557	.501	.479	.475	.564	.472	.325
.5	1	1	.05	.453	.453	.440	.418	.415	.505	.438	.308
			.10	.606	.606	.599	.571	.570	.648	.608	.444
1	1	1	.05	.466	.463	.470	.475	.478	.388	.367	.271
			.10	.635	.635	.638	.641	.640	.535	.545	.407
0	1	0	.05	.358	.096	.271	.233	.242	.502	.269	.290
			.10	.499	.189	.408	.376	.377	.641	.412	.407
0	1	.5	.05	.386	.223	.292	.278	.283	.387	.314	.243
			.10	.525	.366	.437	.426	.423	.534	.478	.369
.5	1	0	.05	.386	.107	.309	.278	.282	.514	.305	.341
			.10	.531	.222	.452	.430	.431	.654	.442	.471
.5	1	.5	.05	.410	.241	.343	.318	.326	.400	.310	.259
			.10	.553	.391	.493	.471	.480	.547	.473	.382
1	0	0	.05	.372	.052	.068	.239	.241	.484	.189	.284
			.10	.514	.121	.147	.377	.376	.642	.280	.402
1	.5	0	.05	.405	.082	.166	.281	.285	.500	.205	.338
			.10	.546	.185	.294	.433	.429	.652	.320	.463
1	.5	.5	.05	.418	.197	.220	.319	.324	.274	.176	.181
			.10	.563	.348	.365	.477	.474	.415	.299	.290
1	1	.5	.05	.466	.262	.382	.406	.414	.274	.296	.306
			.10	.621	.433	.547	.573	.573	.415	.464	.433
					b. Exponential						
0	0	1	.05	.496	.496	.405	.338	.341	.508	.361	.224
			.10	.643	.643	.565	.504	.500	.673	.531	.374
0	.5	1	.05	.560	.560	.480	.429	.431	.728	.600	.432
			.10	.696	.696	.628	.582	.585	.838	.741	.586
.5	.5	1	.05	.608	.608	.541	.498	.501	.632	.508	.362
			.10	.735	.735	.676	.645	.638	.755	.653	.503
.5	1	1	.05	.628	.628	.594	.574	.565	.709	.618	.476
			.10	.753	.753	.726	.709	.711	.813	.749	.609
1	1	1	.05	.612	.612	.614	.612	.611	.520	.462	.349
			.10	.748	.748	.745	.744	.745	.656	.625	.477
0	1	0	.05	.493	.128	.382	.341	.352	.669	.358	.396
			.10	.641	.241	.540	.508	.511	.815	.521	.547
0	1	.5	.05	.560	.353	.427	.428	.427	.538	.387	.325
			.10	.695	.506	.577	.585	.585	.705	.563	.468
.5	1	0	.05	.559	.174	.475	.428	.423	.730	.482	.526
			.10	.695	.311	.624	.583	.581	.843	.629	.655
.5	1	.5	.05	.616	.401	.535	.502	.506	.612	.453	.423
			.10	.736	.552	.672	.648	.642	.753	.620	.560
1	0	0	.05	.513	.067	.092	.341	.345	.663	.254	.403
			.10	.667	.146	.188	.509	.509	.814	.356	.549
1	.5	0	.05	.580	.141	.274	.430	.436	.720	.247	.535
			.10	.718	.280	.431	.589	.589	.838	.383	.665
1	.5	.5	.05	.634	.330	.365	.510	.510	.393	.195	.238
			.10	.750	.495	.516	.656	.653	.574	.327	.375
1	1	.5	.05	.656	.412	.563	.576	.575	.405	.392	.442
			.10	.767	.590	.702	.715	.707	.566	.547	.574

powers for $T_{\hat{p}_m}$ and D reported in Tables 6, 7, and 8 are due to the randomization employed in the study and the fact that only simulated, not exact, critical values were used for the different tests.) This implies, for the test based on $T_{\hat{p}_m}$ with equal sample sizes, that knowing that the treatment effects follow an umbrella pattern under the alternative but without any additional information about the location of the peak group is essentially equivalent to knowing nothing about the pattern of the treatment effects.

Comparing the tests based on T_p, $T_{\hat{p}_s}$, and $T_{\hat{p}_m}$ (or D) with the corresponding tests for umbrella alternatives based on A_p, $S_M(\frac{1}{2})$, and $A_{\hat{p}_m}$, respectively, we have several observations. First, the test based on A_p generally provides a better test for H_A than the one based on T_p. Second, $T_{\hat{p}_s}$ and $T_{\hat{p}_m}$ (or

Table 7

Monte Carlo power estimates for $k = 4$ and $n_0 = n_1 = n_2 = n_3 = n_4 = 5$

$\theta_1 - \theta_0$	$\theta_2 - \theta_0$	$\theta_3 - \theta_0$	$\theta_4 - \theta_0$	α	T_p	S	$T_{\hat{p}_s}$	$T_{\hat{p}_m}$	D	A_p	$S_M(\frac{1}{2})$	$A_{\hat{p}_m}$
						a. Normal						
0	0	0	1	.05	.361	.361	.260	.211	.215	.345	.237	.137
				.10	.507	.507	.394	.336	.338	.496	.360	.228
0	0	.5	1	.05	.384	.384	.290	.246	.245	.497	.383	.227
				.10	.534	.534	.424	.377	.377	.646	.519	.349
0	.5	1	1.5	.05	.673	.673	.562	.524	.515	.841	.758	.573
				.10	.804	.804	.708	.669	.668	.915	.857	.706
.5	1	1	1.5	.05	.703	.703	.620	.589	.583	.752	.656	.465
				.10	.825	.825	.753	.730	.727	.856	.782	.610
1	1	1	1	.05	.487	.487	.492	.493	.490	.348	.345	.244
				.10	.656	.656	.655	.650	.651	.493	.504	.388
0	0	1	0	.05	.345	.103	.231	.212	.211	.450	.261	.221
				.10	.484	.198	.358	.331	.335	.607	.396	.346
0	.5	1	0	.05	.372	.117	.272	.243	.239	.549	.365	.326
				.10	.512	.224	.409	.372	.371	.695	.511	.464
0	.5	1	.5	.05	.391	.242	.283	.279	.278	.468	.394	.283
				.10	.537	.395	.426	.415	.417	.625	.546	.424
.5	.5	1	0	.05	.383	.125	.302	.276	.279	.475	.299	.287
				.10	.525	.243	.440	.413	.416	.624	.441	.437
0	1	1	0	.05	.405	.140	.319	.334	.321	.616	.454	.448
				.10	.561	.269	.491	.471	.470	.755	.612	.609
0	1	0	0	.05	.341	.061	.208	.214	.207	.510	.189	.255
				.10	.484	.134	.317	.331	.327	.665	.299	.400
0	1	.5	.5	.05	.381	.206	.252	.273	.274	.338	.266	.207
				.10	.529	.343	.387	.412	.416	.495	.410	.344
.5	1	.5	0	.05	.387	.106	.290	.273	.273	.525	.271	.346
				.10	.539	.214	.432	.416	.409	.676	.414	.503
1	0	0	0	.05	.360	.053	.070	.222	.213	.441	.124	.216
				.10	.502	.112	.142	.336	.335	.597	.196	.337
1	.5	0	0	.05	.385	.063	.143	.245	.245	.545	.151	.313
				.10	.528	.142	.249	.371	.369	.690	.247	.456
1.5	1	.5	0	.05	.678	.142	.385	.517	.519	.820	.300	.640
				.10	.804	.290	.557	.669	.668	.904	.451	.773
1.5	1	1	.5	.05	.713	.323	.482	.586	.587	.523	.266	.378
				.10	.829	.518	.651	.728	.725	.676	.418	.542
						b. Exponential						
0	0	0	1	.05	.491	.491	.390	.290	.287	.438	.308	.185
				.10	.645	.645	.540	.445	.441	.605	.453	.298
0	0	.5	1	.05	.550	.550	.428	.366	.360	.714	.586	.399
				.10	.697	.697	.579	.517	.513	.837	.725	.543
0	.5	1	1.5	.05	.811	.811	.723	.671	.660	.958	.917	.810
				.10	.897	.897	.830	.792	.787	.981	.962	.885
.5	1	1	1.5	.05	.842	.842	.770	.740	.741	.907	.847	.699
				.10	.915	.915	.862	.837	.842	.956	.914	.797
1	1	1	1	.05	.641	.641	.636	.638	.630	.460	.448	.328
				.10	.764	.764	.756	.754	.750	.610	.591	.470
0	0	1	0	.05	.473	.131	.321	.298	.293	.598	.338	.292
				.10	.629	.243	.474	.446	.440	.769	.497	.449
0	.5	1	0	.05	.530	.173	.403	.359	.356	.782	.571	.520
				.10	.675	.304	.560	.514	.515	.886	.721	.678
0	.5	1	.5	.05	.582	.388	.431	.430	.431	.707	.565	.459
				.10	.715	.554	.579	.578	.579	.832	.719	.621
.5	.5	1	0	.05	.574	.212	.473	.425	.430	.696	.494	.470
				.10	.708	.350	.620	.575	.579	.819	.633	.624
0	1	1	0	.05	.545	.209	.444	.421	.428	.803	.624	.615
				.10	.695	.373	.605	.584	.584	.896	.761	.765
0	1	0	0	.05	.473	.075	.284	.290	.295	.681	.257	.358
				.10	.631	.156	.420	.444	.450	.836	.393	.548
0	1	.5	.5	.05	.573	.327	.370	.429	.436	.480	.340	.272
				.10	.713	.492	.517	.574	.579	.666	.504	.442
.5	1	.5	0	.05	.581	.188	.452	.428	.437	.769	.416	.564
				.10	.718	.326	.604	.573	.578	.878	.573	.719
1	0	0	0	.05	.491	.059	.083	.290	.291	.597	.164	.296
				.10	.645	.124	.167	.445	.442	.758	.247	.458
1	.5	0	0	.05	.548	.094	.225	.361	.363	.781	.198	.524
				.10	.691	.193	.360	.513	.517	.882	.309	.681
1.5	1	.5	0	.05	.813	.245	.545	.669	.667	.945	.307	.833
				.10	.897	.414	.699	.790	.796	.976	.475	.912
1.5	1	1	.5	.05	.849	.489	.642	.742	.739	.738	.276	.546
				.10	.913	.666	.771	.843	.838	.852	.440	.699

Table 8

Monte Carlo power estimates for $k = 5$ and $n_0 = n_1 = n_2 = n_3 = n_4 = n_5 = 5$

$\theta_1 - \theta_0$	$\theta_2 - \theta_0$	$\theta_3 - \theta_0$	$\theta_4 - \theta_0$	$\theta_5 - \theta_0$	α	T_p	S	$T_{\hat{p}_s}$	$T_{\hat{p}_m}$	D	A_p	$S_M(\frac{1}{2})$	$A_{\hat{p}_m}$
a. Normal													
0	0	0	0	1	.05	.383	.383	.261	.218	.209	.311	.206	.123
					.10	.521	.521	.397	.335	.324	.463	.319	.195
0	0	.5	1	1.5	.05	.693	.693	.547	.506	.498	.864	.777	.598
					.10	.808	.808	.704	.651	.650	.933	.869	.709
0	.5	1	1.5	2	.05	.899	.899	.810	.779	.775	.983	.963	.872
					.10	.953	.953	.905	.880	.875	.995	.984	.923
.5	1	1	1.5	2	.05	.917	.917	.845	.826	.820	.948	.903	.777
					.10	.961	.961	.924	.907	.903	.978	.953	.855
1	1	1	1	1	.05	.532	.532	.521	.525	.528	.309	.315	.241
					.10	.684	.684	.683	.685	.686	.463	.473	.366
0	0	0	1	0	.05	.373	.115	.224	.211	.204	.401	.233	.192
					.10	.506	.213	.349	.334	.330	.559	.362	.289
0	0	.5	1	0	.05	.395	.136	.259	.248	.237	.541	.361	.308
					.10	.531	.240	.391	.372	.364	.697	.502	.425
0	.5	1	1.5	0	.05	.678	.238	.533	.505	.500	.873	.736	.702
					.10	.795	.393	.689	.652	.644	.938	.844	.807
.5	.5	1	1.5	.5	.05	.710	.414	.569	.555	.549	.765	.617	.541
					.10	.818	.577	.713	.701	.692	.866	.761	.675
0	0	1	0	0	.05	.370	.071	.184	.271	.211	.480	.192	.246
					.10	.505	.145	.293	.331	.329	.641	.315	.375
0	0	1	.5	0	.05	.394	.112	.217	.245	.241	.491	.287	.304
					.10	.533	.209	.345	.370	.354	.644	.430	.432
0	.5	1	.5	0	.05	.417	.124	.260	.273	.264	.568	.344	.387
					.10	.553	.227	.400	.408	.397	.714	.501	.527
.5	1	1.5	.5	0	.05	.696	.187	.549	.534	.531	.835	.572	.697
					.10	.813	.332	.696	.675	.674	.915	.715	.811
0	1	0	0	0	.05	.380	.055	.170	.217	.213	.488	.129	.250
					.10	.503	.119	.270	.333	.328	.639	.214	.375
0	1	.5	0	0	.05	.401	.072	.216	.249	.243	.574	.201	.351
					.10	.530	.153	.334	.371	.366	.713	.318	.493
1	1.5	1	.5	0	.05	.680	.166	.437	.497	.563	.851	.429	.720
					.10	.794	.298	.597	.648	.707	.923	.600	.831
1	1.5	1	1	.5	.05	.747	.385	.595	.590	.588	.625	.357	.488
					.10	.846	.566	.751	.741	.734	.761	.523	.630
1	0	0	0	0	.05	.384	.049	.069	.218	.212	.402	.076	.194
					.10	.519	.106	.144	.334	.329	.561	.128	.287
1.5	1	.5	0	0	.05	.693	.096	.372	.507	.496	.878	.252	.695
					.10	.809	.210	.538	.654	.642	.941	.387	.802
2	1.5	1	.5	0	.05	.899	.241	.689	.776	.775	.971	.357	.906
					.10	.954	.435	.829	.879	.875	.989	.534	.952
2	1.5	1	1	.5	.05	.915	.435	.719	.821	.812	.833	.250	.661
					.10	.960	.633	.850	.906	.900	.914	.393	.782
b. Exponential													
0	0	0	0	1	.05	.496	.496	.340	.279	.284	.408	.275	.165
					.10	.652	.652	.507	.424	.430	.573	.415	.260
0	0	.5	1	1.5	.05	.802	.802	.681	.626	.632	.972	.928	.836
					.10	.893	.893	.808	.755	.757	.990	.963	.900
0	.5	1	1.5	2	.05	.940	.940	.882	.846	.853	.997	.984	.947
					.10	.973	.973	.943	.920	.920	.999	.989	.961
.5	1	1	1.5	2	.05	.956	.956	.915	.890	.895	.992	.969	.878
					.10	.980	.980	.958	.942	.944	.996	.982	.927
1	1	1	1	1	.05	.661	.661	.646	.665	.661	.427	.412	.324
					.10	.772	.772	.763	.775	.768	.569	.563	.450
0	0	0	1	0	.05	.483	.138	.289	.283	.288	.518	.296	.250
					.10	.632	.240	.437	.430	.433	.694	.451	.364
0	0	.5	1	0	.05	.534	.179	.364	.339	.345	.759	.559	.497
					.10	.675	.297	.520	.484	.491	.878	.712	.630
0	.5	1	1.5	0	.05	.792	.322	.674	.632	.634	.973	.907	.883
					.10	.882	.483	.798	.755	.762	.990	.956	.938
.5	.5	1	1.5	.5	.05	.850	.570	.737	.729	.734	.928	.828	.710
					.10	.917	.704	.841	.823	.831	.971	.908	.808
0	0	1	0	0	.05	.481	.080	.242	.282	.281	.633	.241	.320
					.10	.636	.159	.372	.422	.421	.801	.383	.484
0	0	1	.5	0	.05	.532	.156	.284	.339	.343	.678	.382	.423
					.10	.673	.270	.434	.482	.487	.825	.546	.574

Table 8
Continued

$\theta_1-\theta_0$	$\theta_2-\theta_0$	$\theta_3-\theta_0$	$\theta_4-\theta_0$	$\theta_5-\theta_0$	α	T_p	S	$T_{\hat{p}_s}$	$T_{\hat{p}_m}$	D	A_p	$S_M(\frac{1}{2})$	$A_{\hat{p}_m}$
0	.5	1	.5	0	.05	.579	.192	.376	.394	.400	.806	.517	.609
					.10	.711	.313	.536	.537	.540	.901	.676	.743
.5	1	1.5	.5	0	.05	.828	.292	.710	.683	.687	.964	.785	.880
					.10	.903	.451	.825	.796	.802	.986	.872	.939
0	1	0	0	0	.05	.495	.062	.210	.281	.282	.651	.164	.323
					.10	.634	.128	.327	.423	.432	.807	.265	.484
0	1	.5	0	0	.05	.542	.100	.292	.342	.339	.800	.266	.536
					.10	.675	.192	.437	.484	.486	.898	.405	.691
1	1.5	1	.5	0	.05	.840	.310	.717	.699	.698	.967	.556	.891
					.10	.907	.481	.832	.810	.813	.987	.947	.712
1	1.5	1	1	.5	.05	.865	.536	.741	.719	.720	.836	.448	.673
					.10	.919	.685	.838	.830	.825	.916	.607	.798
1	0	0	0	0	.05	.502	.052	.080	.280	.284	.526	.090	.243
					.10	.654	.114	.163	.421	.432	.702	.149	.366
1.5	1	.5	0	0	.05	.809	.151	.505	.625	.632	.973	.263	.888
					.10	.896	.280	.665	.761	.762	.992	.421	.940
2	1.5	1	.5	0	.05	.944	.358	.785	.848	.847	.996	.344	.973
					.10	.974	.547	.883	.921	.919	.999	.521	.988
2	1.5	1	1	.5	.05	.959	.575	.824	.891	.890	.940	.233	.846
					.10	.981	.733	.903	.943	.940	.962	.373	.912

D) can be viewed as competitors to $S_M(\frac{1}{2})$ and $A_{\hat{p}_m}$, respectively. In particular, when the treatment effects are all greater than the control, the test based on $T_{\hat{p}_m}$ (or D) is superior to the one based on $A_{\hat{p}_m}$. Finally, for the widespread umbrella corresponding to $\theta_0 < \theta_1 = \cdots = \theta_k$, the powers of the tests based on T_p, $T_{\hat{p}_s}$, and $T_{\hat{p}_m}$ (or D) are similar. In this case, they all do better than the tests for umbrella alternatives.

Although tests for umbrella alternatives based on A_p, $S_M(\frac{1}{2})$, and $A_{\hat{p}_m}$ can be used for testing the alternative hypothesis H_A considered in this paper, they only provide single tests. In comparing several umbrella pattern treatment effects with a control, however, experimenters usually prefer procedures that can be used to determine which treatments (if any) are more effective than the control. Therefore, as a direct consequence of the simulation results, we have several recommendations. When the prior information that the treatment effects have an umbrella pattern under the alternative is available, the test based on T_p should be used if one is relatively confident of the location of the peak group. The test based on $T_{\hat{p}_s}$ is recommended if the peak group of the umbrella is unknown, but is believed to be relatively close to the kth population. For the case where no information about the location of the peak group is available, Dunn's test D is suggested since it is computationally less complicated than the test based on $T_{\hat{p}_m}$ and the two procedures are equivalent when the sample sizes are equal.

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Résumé

Cet article traite de la comparaison des effets de traitement disposés "en parapluie" avec un contrôle à une voie. Le problème est de tester s'il y a au moins un traitement qui soit meilleur que la référence. Des tests indépendants des distributions sont proposés dans les deux cas où le sommet du parapluie est connu et inconnu. Les approximations des seuils critiques sont donnés et les résultats d'une étude basée sur les méthodes de Monte Carlo sont discutés.

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